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Oscillation Properties of Third Order Nonlinear Delay Dynamic Equations on Time Scales

F. S. AL-dheleai
Department of Mathematics, Faculty of Education and Applied Sciences,
Amran University, Amran, Yamen

Abstract

In this paper, we shall investigate the oscillatory properties of third order nonlinear delay dynamic equations. Applying suitable comparison theorems and by a Riccati transformation technique, we establish some new sufficient conditions which insure that every solution of this equation either oscillates or converges to zero. Our results not only unify the oscillation of third order nonlinear differential and difference equations but also can be applied to different types of time scales with $sup\mathbb{T} = \infty$. We support our results with suitable examples.

Keywords: - Dynamic equation, Oscillatory solutions, Delay.
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الملخص: في هذه الورقة، سنبحث في الخصائص التذبذبية لمعادلات التأخير غير الخطية من الدرجة الثالثة. من خلال تطبيق نظريات المقارنة المناسبة وتقنية التحويل Riccati، نضع بعض الشروط الجديدة الكافية التي تضمن أن كل حل لهذه المعادلة إما يتذبذب أو يتقارب إلى الصفر. لا تعمل نتائجنا على توحيد تذبذب المعادلات التفاضلية والفرقية غير الخطية من الدرجة الثالثة فحسب، بل يمكن تطبيقها أيضًا على أنواع مختلفة من المقاييس الزمنية. نحن ندعم نتائجنا بأمثلة مناسبة.

1. Introduction

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. The theory of time scales was introduced by Hilger [1] in his Ph.D. thesis in 1988 in order to unify continuous and discrete analysis. Two books on the subject of time scales by Bohner and Peterson [2, 3] and the references cited therein. By comparison with some first dynamic equations whose oscillatory characters are known and by means of a Riccati transformation technique, we obtain several new sufficient conditions for the oscillation for solutions of the nonlinear dynamic equation with Delay of the form

$$\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta} + \sum_{i=1}^{n} q_i(t)\, f\left(x(g_i(t))\right) = 0, \qquad t \geq t_0\,. \tag{1.1}$$

Where α_1, α_2 are quotients of positive odd integers. We assume that the following conditions satisfied:

(A_1) a *and* b are positive real valued rd − continuous functions on $\mathbb{T}$.
(A_2) $q_i \in C_{rd}([0,\infty)_{\mathbb{T}}, [0,\infty))$, for $i = 1,2,\dots,n$.
(A_3) $f \in C(\mathbb{R},\mathbb{R})$ such that $xf(x) > 0$, $f'(x) > 0$ $for\ all\ x \neq 0$.
(A_4) $g_i \in C_{rd}([0,\infty)_{\mathbb{T}}, [0,\infty))$ such that $g_i(t) \leq t$, $g_i^{\Delta}(t) \geq 0$ and $\lim_{t\to\infty} \inf g_i(t) = \infty$, $i = 1,2,\dots,n$.

In addition, we will make use of the following conditions:

(S_1) $-f(-xy) \geq f(xy) \geq f(x)f(y)$ $for\ xy > 0$,
(S_2) $f(x)/x^{\alpha} \geq K > 0$, K is a real constant, $x > 0$,
(S_3) $f(x) - f(y) = B(x,y)(x-y)$ $for\ x,y \neq 0$,

where B is a nonnegative real valued function and $f^{\frac{1}{\alpha}-1}(x)B(x,y) \geq \lambda > 0$ $for\ x,y \neq 0$ $and\ \lambda\ is\ a\ constant$.

If $\mathbb{T} = R$, the equation (1.1) becomes the third order nonlinear delay differential equation of the form

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* E-mail: faisalsaleh69@yahoo.com

$$\left(a(t)\left(\left(b(t)\left(x'(t)\right)^{\alpha_1}\right)'\right)^{\alpha_2}\right)' + \sum_{i=1}^{n} q_i(t)\, f\left(x\left(g_i(t)\right)\right) = 0, \qquad t \ge t_0\,. \tag{1.2}$$

If $\mathbb{T} = \mathbb{N}$, the equation (1.1) becomes the third order nonlinear delay difference equation of the form

$$\Delta\left(a(n)\left(\Delta\left(b(n)\left(\Delta x(n)\right)^{\alpha_1}\right)\right)^{\alpha_2}\right) + \sum_{i=1}^{n} q_i(n)\, f\left(x\left(g_i(n)\right)\right) = 0, \qquad n \ge n_0\,. \tag{1.3}$$

In recent years, there has been an increasing interest in the study of the problem of determining the oscillation and non-oscillation of solutions of dynamic equations of the equation (1.1) and its special cases. In (2006) Erbe et al. proved several theorems provided sufficient conditions for oscillation of all solutions of the third order dynamic equation of the form:

$$\left(c(t)\left(\left(a(t)\left(x(t)\right)^{\Delta}\right)^{\Delta}\right)^{\gamma}\right)^{\Delta} + f\left(t, x(t)\right) = 0, \tag{1.4}$$

depend on condition

$$\int_{t_0}^{\infty} c^{\frac{-1}{\gamma}}(s)\Delta s = \infty, \int_{t_0}^{\infty} a^{-1}(s)\, \Delta s = \infty. \tag{1.5}$$

In (2011)by means of the Riccati transformation technique, Li et al. studied the Oscillation criteria for third-order nonlinear dynamic equations

$$x^{\Delta^3}(t) + p(t)x^{\gamma}\left(\tau(t)\right) = 0. \tag{1.6}$$

And by condition (1.5) they discussed the oscillation results for the third order nonlinear delay dynamic equations

$$\left(c(t)\left(\left(a(t)\left(x(t)\right)^{\Delta}\right)^{\Delta}\right)^{\gamma}\right)^{\Delta} + f\left(t, x\left(\tau(t)\right)\right) = 0, \tag{1.7}$$

In (2011) by a Riccati transformation technique, Han et al. established some sufficient conditions for the oscillation behavior of solution of third-order nonlinear delay dynamic equations of the form:

$$\left(a(t)\left(b(t)\left(x(t)\right)^{\Delta}\right)^{\Delta}\right)^{\Delta} + q(t) f\left(x\left(\tau(t)\right)\right) = 0, \tag{1.8}$$

under the condition

$$\int_{t_0}^{\infty} a^{-1}(s)\Delta s = \infty, \int_{t_0}^{\infty} b^{-1}(s)\, \Delta s = \infty. \tag{1.9}$$

In (2011) by a Riccati transformation technique, Li et al. studied the Oscillation results for third-order nonlinear delay dynamic equations on time scales of the form

$$\left(a(t)\left(r(t)\left(x^{\Delta}(t)\right)^{\Delta}\right)^{\gamma}\right)^{\Delta} + f\left(t, x\left(\tau(t)\right)\right) = 0, \tag{1.10}$$

under the condition (1.5).

In (2021) by a Riccati transformation technique, AL-dheleai et al. discussed the oscillation criteria for third order nonlinear mixed neutral dynamic equations of the form

$$\left(a(t)\left(b(t)\left(x(t) + p_1(t)x(t-\tau_1) + p_2(t)x(t+\tau_2)\right)^{\Delta}\right)^{\Delta}\right)^{\Delta} + q_1(t)x(t-\tau_3) + q_2(t)x(t+\tau_4) = 0\,, \qquad t \ge t_0. \tag{1.11}$$

under the condition (1.9).

For an excellent introduction to the calculus on time scales, see Hilger [1], and Bohner and Peterson[2,3]. For further results concerning the oscillatory and asymptotic behavior of third order dynamic equation, we refer to the papers [4-19] and the references cited therein. Since we are interested in the oscillatory behavior of solutions near infinity, we assume that $sup\mathbb{T} = \infty$ (unbounded above) and define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} \coloneqq [t_0, \infty) \cap \mathbb{T}$. By a solution of the equation (1.1), we mean a nontrivial real-valued function $x \in C^1_{rd}[\mathbb{T}_x, \infty), \mathbb{T}_x \geq t_0$ which satisfies equation (1.1) on the $[\mathbb{T}_x, \infty)$, where C_{rd} is the space of rd −continuous functions. A solution x of the equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and non-oscillatory otherwise. Equation (1.1) is called oscillatory if all its solutions are oscillatory. The main aim of this paper is to establish some sufficient conditions which guarantee that the equation (1.1) has oscillatory solutions or the solutions tend to zero as $n \to \infty$. In this paper, the details of the proofs of results for non-oscillatory solutions will be carried out only for eventually positive solutions, since the arguments are similar for eventually negative solutions. We provide some examples to illustrate the main results. Our results not only unify the oscillation of third order nonlinear differential and difference equations but also can be applied to different types of time scales with $sup\mathbb{T} = \infty$. The paper is organized as follows. In section 2, we will state and prove the main oscillation theorems. In section 3, we illustrate our results with suitable examples.

2. Main Results

In this section, we establish some new oscillation criteria for the equation (1.1) under the following conditions:

$$\int_{t_0}^{\infty} a^{-\frac{1}{\alpha_2}}(s)\Delta s = \infty, \int_{t_0}^{\infty} b^{-\frac{1}{\alpha_1}}(s)\,\Delta s = \infty. \tag{2.1}$$

$$\int_{t_0}^{\infty} a^{-\frac{1}{\alpha_2}}(s)\Delta s < \infty, \int_{t_0}^{\infty} b^{-\frac{1}{\alpha_1}}(s)\,\Delta s = \infty. \tag{2.2}$$

$$\int_{t_0}^{\infty} a^{-\frac{1}{\alpha_2}}(s)\Delta s < \infty, \int_{t_0}^{\infty} b^{-\frac{1}{\alpha_1}}(s)\,\Delta s < \infty. \tag{2.3}$$

We begin with some useful lemmas, which will be used in obtaining our main results. We Let

$$g(t) = min\{g_1(t), g_2(t), \dots, g_n(t)\}, \;\; Q(t) = \sum_{i=1}^{n} q_i(t),$$

$$\delta_1(g(t), t_2) = \int_{t_2}^{g(t)} a^{-\frac{1}{\alpha_2}}(s)\,\Delta s, \delta(t) = \int_{t}^{\infty} a^{-\frac{1}{\alpha_2}}(s)\,\Delta s\,, \Psi(t) = KQ(t)\left(\int_{t_2}^{g(t)} b^{-\frac{1}{\alpha_1}}(s)\,\Delta s\right)^{\alpha}.$$

Lemma 2.1. Let $x(t)$ be an eventually positive solution of the equation (1.1) which satisfies

$$x^{\Delta}(t) > 0\,, \left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta} > 0\,, \left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta} \leq 0 \;\; for\ all\ t \geq t_0.$$

Then there exists $t \geq t_2$ such that

$$x^{\Delta}(t) \geq b^{-\frac{1}{\alpha_1}}(t)\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\frac{1}{\alpha}}\left(\int_{t_2}^{t} a^{-\frac{1}{\alpha_2}}(s)\,\Delta s\right)^{\frac{1}{\alpha_1}}, \tag{2.4}$$

where

$\alpha := \alpha_1\alpha_2$.

Proof. Since $\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta} \leq 0$, we have $a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}$ is non-increasing; then, we obtain,

$$b(t)\left(x^{\Delta}(t)\right)^{\alpha_1} = b(t_2)\left(x^{\Delta}(t_2)\right)^{\alpha_1} + \int_{t_2}^{t} a^{-\frac{1}{\alpha_2}}(s)\left(a(s)\left(\left(b(s)\left(x^{\Delta}(s)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\frac{1}{\alpha_2}} \Delta s$$

$$\geq \left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\frac{1}{\alpha_2}} \int_{t_2}^{t} a^{-\frac{1}{\alpha_2}}(s)\,\Delta s.$$

It follows that

$$x^{\Delta}(t) \geq b^{-\frac{1}{\alpha_1}}(t)\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\frac{1}{\alpha_1\alpha_2}}\left(\int_{t_2}^{t} a^{-\frac{1}{\alpha_2}}(s)\,\Delta s\right)^{\frac{1}{\alpha_1}}.$$

The proof is complete. ■

Lemma 2.2. Assuming that (2.1) holds, let $x(t)$be an eventually positive solution of equation (1.1). Then, for sufficiently large t, there are only two possible cases:

(I): $x(t) > 0,\ x^{\Delta}(t) > 0\,,\ \left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta} > 0\,,$

(II): $x(t) > 0,\ x^{\Delta}(t) < 0\,,\ \left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta} > 0\,.$

Proof. Pick $t_1 \geq t_0$ such that $x(g(t)) > 0$, for $t \geq t_1$. Since $x(t)$ is an eventually positive solution of (1.1). From equation (1.1), $(A_1), (A_1)$ *and* (A_3), we see that

$$\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta} \leq 0, for\ all\ t \geq t_1.$$

Then, $a(t)\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}$ is a non-increasing function and thus $x(t)$, $x^{\Delta}(t)$ and $\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}$
are eventually of one sign. There are the following four possibilities to consider

(I): $x^{\Delta}(t) > 0\,,\ \left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta} > 0\ for\ all\ large\ t,$

(II): $x^{\Delta}(t) < 0\,,\ \left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta} > 0\ for\ all\ large\ t,$

(III): $x^{\Delta}(t) > 0\,,\ \left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta} < 0\ for\ all\ large\ t,$ and

(IV): $x^{\Delta}(t) < 0\,,\ \left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta} < 0\ for\ all\ large\ t.$

We claim that $\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta} > 0$. If not, then, $b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}$ is strictly decreasing there exists a negative constant M and $t_3 \geq t_2$ such that

$$a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2} < M\ for\ all\ t \geq t_3.$$

Dividing by $a(t)$ and integrating the above inequality from t_3 *to* t , we obtain

$$b(t)\left(x^{\Delta}(t)\right)^{\alpha_1} \leq b(t_3)\left(x^{\Delta}(t_3)\right)^{\alpha_1} + M^{\frac{1}{\alpha_2}}\int_{t_3}^{t}\left(a(s)\right)^{-\frac{1}{\alpha_2}}\Delta s.$$

Letting $t \to \infty$, and using (2.1) then $b(t)\left(x^{\Delta}(t)\right)^{\alpha_1} \to -\infty$. Then there exists a $t_4 \geq t_3$ and constant $K < 0$

$$b(t)\left(x^{\Delta}(t)\right)^{\alpha_1} \leq b(t_4)\left(x^{\Delta}(t_4)\right)^{\alpha_1} = K < 0.$$

Dividing by $b(t)$ and integrating the above inequality from t_4 *to* t, we obtain

$$x(t) \leq x(t_4) + K^{\frac{1}{\alpha_1}}\int_{t_4}^{t}\left(b(s)\right)^{-\frac{1}{\alpha_1}}\Delta s.$$

Letting $t \to \infty$, and using (2.1) then $x(t) \to -\infty$, which contradicts the fact that $x(t) > 0$. Then, we have

$\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta} > 0$. And thus either $x^{\Delta}(t) > 0$ *or* $x^{\Delta}(t) < 0$. The proof is complete.■

Lemma 2.3. Assume that (2.1) and (II) of Lemma 2.2 hold, function $x(t)$ is an eventually positive solution of the equation (1.1). If

$$\int_{t_0}^{\infty}\left(b^{-\frac{1}{\alpha_1}}(v)\left(\int_t^{\infty} a^{-\frac{1}{\alpha_2}}(u)\left(\int_t^{\infty} Q(s)\,\Delta s\right)^{\frac{1}{\alpha_2}}\Delta u\right)^{\frac{1}{\alpha_1}}\right)\Delta v = \infty. \tag{2.5}$$

Then $x(t) \to 0$ *as* $t \to \infty$.

Proof. Pick $t_1 \geq t_0$ such as that $x(g(t)) > 0$. Since $x(t)$ is a positive decreasing solution of equation (1.1), then $\lim_{t\to\infty} x(t) = l_1 \geq 0$. Assume that $l_1 > 0$ then $x(g_i(t)) \geq l_1$ for $t \geq t_2 \geq t_1$. From equation (1.1), we have

$$0 \geq \left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta} + f(l_1)Q(t). \tag{2.6}$$

By integrating equation (2.6) from t *to* ∞, we obtain

$$a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2} \geq f(l_1)\int_t^{\infty} Q(s)\,\Delta s.$$

It follows that

$$\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta} \geq \left(\frac{f(l_1)}{a(t)}\right)^{\frac{1}{\alpha_2}}\left(\int_t^{\infty} Q(s)\,\Delta s\right)^{\frac{1}{\alpha_2}}. \tag{2.7}$$

Integrating the above inequality from t *to* ∞, we find

$$-x^{\Delta}(t) \geq \frac{f^{\frac{1}{\alpha}}(l_1)}{b^{\frac{1}{\alpha_1}}(t)}\left(\int_t^{\infty} a^{-\frac{1}{\alpha_2}}(u)\left(\int_t^{\infty} Q(s)\,\Delta s\right)^{\frac{1}{\alpha_2}}\Delta u\right)^{\frac{1}{\alpha_1}}. \tag{2.8}$$

Integrating the above inequality from t_2 *to* ∞, we find

$$x(t_2) \geq f^{\frac{1}{\alpha}}(l_1)\int_{t_2}^{\infty}\left(b^{-\frac{1}{\alpha_1}}(v)\left(\int_t^{\infty} a^{-\frac{1}{\alpha_2}}(u)\left(\int_t^{\infty} Q(s)\,\Delta s\right)^{\frac{1}{\alpha_2}}\Delta u\right)^{\frac{1}{\alpha_1}}\right)\Delta v.$$

This contradicts condition (2.5). Then $\lim_{t\to\infty} x(t) = 0$. ■

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2.1. Nonexistence of solutions of type (I)

Next, we shall establish some criteria for the nonexistence of solution of type (I) for the equation (1.1).

Theorem 2.1. Let $(A_1)-(A_4)$ and (S_1) hold. If the first order delay equation

$$y^{\Delta}(t)+Q(t)f\left(y^{\frac{1}{\alpha}}(g(t))\right)f\left(\int_{t_0}^{g(t)} b^{-\frac{1}{\alpha_1}}(s)\left(\int_{u=t_0}^{s} a^{-\frac{1}{\alpha_2}}(u)\,\Delta u\right)^{\frac{1}{\alpha_1}}\Delta s\right)=0, \qquad (2.9)$$

is oscillatory, then equation (1.1) has no solution of type (I).

Proof. Let $x(t)$ be an eventually positive solution of equation (1.1) of type (I), then, there is a $t_0\in[t_0,\infty)_{\mathbb{T}}$ such that (I) holds for $t\geq t_0$. From Lemma (2.1), we have

$$x^{\Delta}(t)\geq b^{-\frac{1}{\alpha_1}}(t)y^{\frac{1}{\alpha}}(t)\left(\int_{t_2}^{t} a^{-\frac{1}{\alpha_2}}(s)\,\Delta s\right)^{\frac{1}{\alpha_1}},$$

where $y(t)=a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}$. Integrating the above inequality from t_2 *to* t, we obtain,

$$x(t)\geq\int_{t_2}^{t} b^{-\frac{1}{\alpha_1}}(s)y^{\frac{1}{\alpha}}(s)\left(\int_{t_2}^{s} a^{-\frac{1}{\alpha_2}}(u)\,\Delta u\right)^{\frac{1}{\alpha_1}}\Delta s$$

$$\geq y^{\frac{1}{\alpha}}(t)\int_{t_2}^{t} b^{-\frac{1}{\alpha_1}}(s)\left(\int_{t_2}^{s} a^{-\frac{1}{\alpha_2}}(u)\,\Delta u\right)^{\frac{1}{\alpha_1}}\Delta s.$$

There exists a $t_3\geq t_2$ with $g(t)\geq t_2$ for all $t\geq t_3$, such as that

$$x(g(t))\geq y^{\frac{1}{\alpha}}(g(t))\int_{t_2}^{g(t)} b^{-\frac{1}{\alpha_1}}(s)\left(\int_{t_2}^{s} a^{-\frac{1}{\alpha_2}}(u)\,\Delta u\right)^{\frac{1}{\alpha_1}}\Delta s.$$

From equation (1.1), (S_1) and the last inequality, for $t\geq t_3$, we obtain

$$-y^{\Delta}(t)\geq f\left(x(g(t))\right)Q(t)$$

$$\geq Q(t)f\left(y^{\frac{1}{\alpha}}(g(t))\right)f\left(\int_{t_2}^{g(t)} b^{-\frac{1}{\alpha_1}}(s)\left(\int_{t_2}^{s} a^{-\frac{1}{\alpha_2}}(u)\,\Delta u\right)^{\frac{1}{\alpha_1}}\Delta s\right).$$

Integrating the last inequality from n to ∞, we get

$$y(t)\geq\int_{t_2}^{\infty} Q(s)f\left(y^{\frac{1}{\alpha}}(g(s))\right)f\left(\int_{t_2}^{g(s)} b^{-\frac{1}{\alpha_1}}(v)\left(\int_{t_2}^{v} a^{-\frac{1}{\alpha_2}}(u)\,\Delta u\right)^{\frac{1}{\alpha_1}}\Delta v\right)\Delta s.$$

The function $y(t)$ is obviously strictly decreasing. Hence, by the discrete analog of Theorem 1 in [4], we conclude that there exists a positive solution $y(t)$ of the equation (2.9) which tends to zero. This contradicts that (2.9) is oscillatory. The proof is complete . ■

Corollary 2.1. If $\mathbb{T} = \mathbb{N}$, then (2.9) becomes

$$\Delta(y_n) + Q(n) f\left(y^{\frac{1}{\alpha}}(g(n))\right) f\left(\sum_{s=n_0}^{g(n)-1} b^{-\frac{1}{\alpha_1}}(s)\left(\sum_{u=n_0}^{s-1} a^{-\frac{1}{\alpha_2}}(u)\right)^{\frac{1}{\alpha_1}}\right) = 0,$$

is oscillatory, then equation (1.2) has no solution of type (I).

Corollary 2.2. Let $(A_1) - (A_4)$ and (S_2) hold. If the first order delay equation

$$y^{\Delta}(t) + KQ(t)y(g(t))\left(\int_{t_0}^{g(t)} b^{-\frac{1}{\alpha_1}}(s)\left(\int_{t_0}^{s} a^{-\frac{1}{\alpha_2}}(u)\,\Delta u\right)^{\frac{1}{\alpha_1}}\Delta s\right)^{\alpha} = 0,$$

is oscillatory, then equation (1.1) has no solution of the type (I).

Theorem 2.2. Let $(A_1) - (A_4)$ and (S_3) hold. Further, assume that there exists a positive rd-continuous $\Delta-$ differentiable function $\beta(t)$, such that

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left(\beta(s)Q(s) + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{\left(\beta^{\Delta}(s)\right)_+^{\alpha+1} b^{\alpha_2}(g(s))}{\left(\lambda\beta(s)\delta_1^{\frac{1}{\alpha_1}}(g(s),t_2)\right)^{\alpha}}\right) = \infty, \tag{2.10}$$

$$\left(\beta^{\Delta}(t)\right)_+ := max\{0, \beta^{\Delta}(t)\}.$$

Then equation (1.1) has no solution of the type (I).

Proof. Let $x(t)$ be an eventually positive solution of equation (1.1) of type (I). Then, there is a $t_0 \in [t_0,\infty)_{\mathbb{T}}$ such that (I) holds for $t \geq t_0$. Define the Riccati type function $\omega(t)$ by

$$\omega(t) := \beta(t)\frac{a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}}{f\left(x(g(t))\right)}. \tag{2.11}$$

Then $\omega(t) > 0$. From (2.11) and (S_3), we have

$$\omega^{\Delta}(t) = \beta^{\Delta}(t)\frac{a^{\sigma}(((b(x^{\Delta})^{\alpha_1})^{\Delta})^{\sigma})^{\alpha_2}}{f(x(g^{\sigma}))} + \beta(t)\frac{\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta}}{f\left(x(g(t))\right)}$$

$$-\beta(t)\frac{a((b(x^{\Delta})^{\alpha_1})^{\Delta})^{\alpha_2}\left(f\left(x(g(t))\right)\right)^{\Delta}}{f(x(g^{\sigma}))f\left(x(g(t))\right)}.$$

$$\leq \beta^{\Delta}(t)\frac{a^{\sigma}(((b(x^{\Delta})^{\alpha_1})^{\Delta})^{\sigma})^{\alpha_2}}{f(x(g^{\sigma}))}+\beta(t)\frac{\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta}}{f\left(x(g(t))\right)}$$
$$-\beta(t)\frac{a^{\sigma}(((b(x^{\Delta})^{\alpha_1})^{\Delta})^{\sigma})^{\alpha_2}B\left((x(g^{\sigma})),x(g(t))\right)}{f(x(g^{\sigma}))f\left(x(g(t))\right)}\left(x(g(t))\right)^{\Delta}. \quad (2.12)$$

By Lemma2.1, there exists $t_3 \geq t_2$ with $g(t) \geq t_2$ for all $t \geq t_3$ such that

$$\left(x(g(t))\right)^{\Delta}$$
$$\geq b^{-\frac{1}{\alpha_1}}(g(t))\left(a(g(t))\left(\left(b(g(t))\left(x^{\Delta}(g(t))\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\frac{1}{\alpha}}\left(\int_{t_2}^{g(t)} a^{-\frac{1}{\alpha_2}}(s)\,\Delta s\right)^{\frac{1}{\alpha_1}}. \quad (2.13)$$

Since $\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta} \leq 0, g(t) < t$, we get

$$a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}$$
$$\leq a(g(t))\left(\left(b(g(t))\left(x^{\Delta}(g(t))\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}. \quad (2.14)$$

Then it follows that

$$a^{\sigma}(((b(x^{\Delta})^{\alpha_1})^{\Delta})^{\sigma})^{\alpha_2}$$
$$\leq a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}. \quad (2.15)$$

It follows from (2.12) that

$$\omega^{\Delta}(t) \leq \frac{\beta^{\Delta}(t)}{\beta^{\sigma}}\omega^{\sigma}(t)+\beta(t)\frac{\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta}}{f\left(x(g(t))\right)}$$
$$-\beta(t)\frac{a^{\sigma}(((b(x^{\Delta})^{\alpha_1})^{\Delta})^{\sigma})^{\alpha_2}B\left((x(g^{\sigma})),x(g(t))\right)}{f(x(g^{\sigma}))f\left(x(g(t))\right)}(a^{\sigma}(((b(x^{\Delta})^{\alpha_1})^{\Delta})^{\sigma})^{\alpha_2})^{\frac{1}{\alpha}}b^{-\frac{1}{\alpha_1}}(g(t))\delta_1^{\frac{1}{\alpha_1}}(g(t),t_2). \quad (2.16)$$

From (1.1), (2.11), (2.16) and (S_3), we have

$$\omega^{\Delta}(t) \leq -\beta(t)Q(t)+\frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t)$$
$$-\lambda\beta(t)\frac{\delta_1^{\frac{1}{\alpha_1}}(g(t),t_2)b^{-\frac{1}{\alpha_1}}(g(t))}{\left(\beta^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}}\left(\omega^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}. \quad (2.17)$$

Using (2.17) and the inequality

$$Bu - Au^{\alpha+1/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}, A > 0, \quad (2.18)$$

we have

$$\omega^{\Delta}(t) \le -\beta(t)Q(t) + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{\left(\beta^{\Delta}(t)\right)^{\alpha+1} b^{\alpha_2}\left(g(t)\right)}{\left(\lambda\beta(t)\delta_1^{\frac{1}{\alpha_1}}(g(t),t_2)\right)^{\alpha}}.$$

Integrating the last inequality from t_2 to t, we obtain

$$\int_{t_2}^{t} \left(\beta(s)Q(s) + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{\left(\beta^{\Delta}(s)\right)_+^{\alpha+1} b^{\alpha_2}\left(g(s)\right)}{\left(\lambda\beta(s)\delta_1^{\frac{1}{\alpha_1}}(g(s),t_2)\right)^{\alpha}} \right) \Delta s \le \omega(t_2).$$

Which is contrary to (2.10). The proof is completed. ∎

Corollary 2.3. If $\mathbb{T} = \mathbb{N}$, then the equation (2.10) becomes

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left(\rho(s)Q(s) - \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{\left(\Delta\rho(s)\right)^{\alpha+1} b^{\alpha_2}\left(g(s)\right)}{\left(\mu\rho(s)\delta_1^{\frac{1}{\alpha_1}}(g(s),n_2)\right)^{\alpha}} \right) = \infty.$$

Then equation (1.2) has no solution of the type (I).

The following theorem gives a Philos -type oscillation criteria for the equation (1.1).

Theorem 2.3. Assume that $(A_1)-(A_4)$ and (S_3) hold. Let $\beta(t)$ be a positive rd-continuous $\Delta-$differentiable function. Furthermore, we assume that there exists a double function $\{H(t,s) \mid t \ge s \ge 0\}$ and $h(t,s)$ such that

(i) $H(t,t) = 0 \; for \; t \ge 0$,

$(ii) H(t,s) > 0 \; for \; t > s > 0$,

(iii) H has a nonpositive continuous $\Delta-$partial derivative $H^{\Delta_s}(t,s)$ with respect to the second variable, and satisfies

$$h(t,s) = -\frac{H^{\Delta_s}(t,s)}{\sqrt{H(t,s)}}.$$

If

$$\limsup_{t\to\infty} \frac{1}{H(t,0)} \int_0^t \left[H(t,s)\beta(s)Q(s) - \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{H(t,s)\vartheta^{\alpha+1}(t,s)}{\left(\varphi(s)\right)^{\alpha}} \right] \Delta s = \infty, \tag{2.19}$$

where

$$\varphi(t) := \frac{\lambda\beta(t)}{(\beta^{\sigma})^{\frac{\alpha+1}{\alpha}}(t) b^{\frac{1}{\alpha_1}}\left(g(t)\right)} \delta_1^{\frac{1}{\alpha_1}}(g(t),t_2), \quad \vartheta(t,s) = \left(\frac{\left(\beta^{\Delta}(t)\right)_+}{\beta^{\sigma}(t)} - \frac{h(t,s)}{\sqrt{H(t,s)}} \right).$$

Then the equation (1.1) has no solution of the type (I).

Proof. Let $x(t)$ be an eventually positive solution of equation (1.1) of the type (I). Then, there is a $t_0 \in [t_0,\infty)_{\mathbb{T}}$ such that (I) holds for $t \ge t_0$. From the proof of Theorem 2.2, we find that (2.17) holds for all $t \ge t_2$. From (2.17), we have

$$\beta(t)Q(t) \le -\omega^{\Delta}(t) + \frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t)$$

$$-\lambda\beta(t)\frac{\delta_1^{\frac{1}{\alpha_1}}(g(t),t_2)b^{-\frac{1}{\alpha_1}}(g(t))}{\left(\beta^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}}\left(\omega^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}. \qquad (2.20)$$

Therefore, we have

$$\int_{t_2}^{t} H(t,s)\beta(s)Q(s)\,\Delta s \le -\int_{t_2}^{t} H(t,s)\omega^{\Delta}(s)\,\Delta s + \int_{t_2}^{t} H(t,s)\frac{\left(\beta^{\Delta}(s)\right)_+}{\beta^{\sigma}(s)}\omega^{\sigma}(s)\,\Delta s$$

$$-\int_{t_2}^{t} H(t,s)\varphi(s)\left(\omega^{\sigma}(s)\right)^{\frac{\alpha+1}{\alpha}}(t)\,\Delta s.$$

Integrating by parts and using $H(t,t) = 0$, we have

$$\int_{t_2}^{t} H(t,s)\beta(s)Q(s)\,\Delta s \le H(t,t_2)\omega(t_2) + \int_{t_2}^{t} H^{\Delta_s}(t,s)\omega^{\sigma}(s)\,\Delta s + \int_{t_2}^{t} H(t,s)\frac{\left(\beta^{\Delta}(s)\right)_+}{\beta^{\sigma}(s)}\omega^{\sigma}(s)\,\Delta s$$

$$-\int_{t_2}^{t} H(t,s)\varphi(s)\left(\omega^{\sigma}(s)\right)^{\frac{\alpha+1}{\alpha}}\,\Delta s$$

$$= H(t,t_2)\omega(t_2) + \int_{t_2}^{t}\left(H^{\Delta_s}(t,s) + H(t,s)\frac{\left(\beta^{\Delta}(s)\right)_+}{\beta^{\sigma}(s)}\right)\omega^{\sigma}(s)\Delta s - \int_{t_2}^{t} H(t,s)\varphi(s)\left(\omega^{\sigma}(s)\right)^{\frac{\alpha+1}{\alpha}}\,\Delta s$$

$$= H(t,t_2)\omega(t_2) + \int_{t_2}^{t} H(t,s)\vartheta(t,s)\,\omega^{\sigma}(s)\Delta s - \int_{t_2}^{t} H(t,s)\varphi(s)\left(\omega^{\sigma}(s)\right)^{\frac{\alpha+1}{\alpha}}\,\Delta s.$$

From (2.18), we have

$$\int_{t_2}^{t} H(t,s)\beta(s)Q(s)\,\Delta s \le H(t,t_2)\omega(t_2) + \int_{t_2}^{t}\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{H(t,s)\vartheta^{\alpha+1}(t,s)}{\left(\varphi(s)\right)^{\alpha}}\,\Delta s.$$

Then,

$$\int_{t_2}^{t}\left[H(t,s)\beta(s)Q(s) - \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{H(t,s)\vartheta^{\alpha+1}(t,s)}{\left(\varphi(s)\right)^{\alpha}}\right]\Delta s \le H(t,t_2)\omega(t_2) \le H(t,0)|\omega(t_2)|.$$

Hence,

$$\int_{0}^{t}\left[H(t,s)\beta(s)Q(s) - \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{H(t,s)\vartheta^{\alpha+1}(t,s)}{\left(\varphi(s)\right)^{\alpha}}\right]\Delta s \le H(t,0)\left\{\int_{0}^{t}|\beta(s)Q(s)|\Delta s + |\omega(t_2)|\right\}.$$

Hence,

$$\limsup_{t\to\infty}\frac{1}{H(t,0)}\int_{0}^{t}\left[H(t,s)\beta(s)Q(s) - \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{H(t,s)\vartheta^{\alpha+1}(t,s)}{\left(\varphi(s)\right)^{\alpha}}\right]\Delta s$$

$$\le \int_{0}^{t}|\beta(s)Q(s)|\Delta s + |\omega(t_2)| < \infty,$$

which is contrary to (2.19). This completes the proof of Theorem 2.3. ■

Corollary 2.4. If $\mathbb{T} = \mathbb{N}$, then (2.19) becomes

$$\lim_{m\to\infty}\sup\frac{1}{H(m,0)}\sum_{n=0}^{m-1}\left(H(m,n)\rho(n)Q(n)-\lambda\frac{\vartheta^{\alpha+1}(m,n)H(m,n)}{(\varphi(n))^{\alpha}}\right)=\infty, \quad (2.19)$$

Then equation (1.2) has no solution of the type (I).

The following theorem gives a Kamenev-type oscillation criteria for equation (1.1).

Theorem 2.4. Let $(A_1)-(A_4)$ and (S_2) hold. Further, assume that there exists a positive rd-continuous $\Delta-$ differentiable non-decreasing function $\beta(t)$, such that

$$\lim_{t\to\infty}\sup\frac{1}{t^r}\int_{t_2}^{t}(t-s)^r\left(K\beta(s)Q(s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left(\beta^{\Delta}(s)\right)_+^{\alpha+1}b^{\alpha_2}(g(s))}{\left(\beta(s)\right)^{\alpha}\delta_1^{\alpha_2}(g(s),t_2)}\right)\Delta s$$
$$=\infty. \quad (2.21)$$

Then equation (1.1) has no solution of type (I).

Proof. Let $x(t)$ be an eventually positive solution of equation (1.1) of type (I). Then, there is a $t_0\in[t_0,\infty)_{\mathbb{T}}$ such that (I) holds for $t\geq t_0$. Define the function $\omega(t)$ by

$$\omega(t) := \beta(t)\frac{a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}}{x^{\alpha}(g(t))}. \quad (2.22)$$

Then $\omega(t)>0$. From (2.22), we have

$$\omega^{\Delta}(t)\leq\beta^{\Delta}(t)\frac{a^{\sigma}(((b(x^{\Delta})^{\alpha_1})^{\Delta})^{\sigma})^{\alpha_2}}{x^{\alpha}(g^{\sigma}(t))}+\beta(t)\frac{\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta}}{x^{\alpha}(g(t))}$$
$$-\beta(t)\frac{a^{\sigma}(((b(x^{\Delta})^{\alpha_1})^{\Delta})^{\sigma})^{\alpha_2}\left(x^{\alpha}(g(t))\right)^{\Delta}}{x^{\alpha}(g^{\sigma}(t))x^{\alpha}(g(t))}.$$

By Lemma 2.2, we see that $x^{\alpha}(g^{\sigma}(t))\geq x^{\alpha}(g(t))$, and that from Keller,s chain rule [2] , we obtain
Thus

$$\omega^{\Delta}(t)\leq\beta^{\Delta}(t)\frac{a^{\sigma}(((b(x^{\Delta})^{\alpha_1})^{\Delta})^{\sigma})^{\alpha_2}}{x^{\alpha}(g^{\sigma}(t))}+\beta(t)\frac{\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta}}{x^{\alpha}(g(t))}$$
$$-\alpha\beta(t)\frac{a^{\sigma}(((b(x^{\Delta})^{\alpha_1})^{\Delta})^{\sigma})^{\alpha_2}\left(x^{\alpha}(g(t))\right)^{\Delta}}{x^{\alpha}(g^{\sigma}(t))x^{\alpha}(g(t))}\left(x(g^{\sigma}(t))\right)^{\alpha-1}x^{\Delta}(g(t)). \quad (2.23)$$

By Lemma2.1, there exists $t_3\geq t_2$ with $g(t)\geq t_2$ for all $t\geq t_3$ such that

$$\left(x(g(t))\right)^{\Delta}\geq b^{-\frac{1}{\alpha_1}}(g(t))\left(a(g(t))\left(\left(b(g(t))\left(x^{\Delta}(g(t))\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\frac{1}{\alpha}}\left(\int_{t_2}^{g(t)}a^{-\frac{1}{\alpha_2}}(s)\,\Delta s\right)^{\frac{1}{\alpha_1}}.$$

Since $\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta}\leq 0\ and\ g(t)<t$, we get

$$a^{\sigma}(((b(x^{\Delta})^{\alpha_1})^{\Delta})^{\sigma})^{\alpha_2} \le a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}$$
$$\le a(g(t))\left(\left(b(g(t))\left(x^{\Delta}(g(t))\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}$$

From (2.23) and the above inequality, we obtain

$$\omega^{\Delta}(t) \le \frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t) + \beta(t)\frac{\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta}}{x^{\alpha}(g(t))}$$
$$-\alpha\frac{\beta(t)}{\left(\beta^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}} b^{\frac{1}{\alpha_1}}(g(t))}\delta_1^{\frac{1}{\alpha_1}}(g(t),t_2)\left(\omega^{\sigma}(t)\right)^{\frac{\alpha+1}{\alpha}}. \quad (2.24)$$

From (1.1),(2.18), (2.24) and (S_2), we have

$$\omega^{\Delta}(t) \le -K\beta(t)Q(t)$$
$$+\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left(\beta^{\Delta}(t)\right)_+^{\alpha+1} b^{\alpha_2}(g(t))}{\left(\beta(t)\right)^{\alpha}\delta_1^{\alpha_2}(g(t),t_2)}. \quad (2.25)$$

From (2.25) for $t \ge t_2$, we obtain

$$\int_{t_2}^{t}(t-s)^r\left(K\beta(s)Q(s) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left(\beta^{\Delta}(s)\right)_+^{\alpha+1} b^{\alpha_2}(g(s))}{\left(\beta(s)\right)^{\alpha}\delta_1^{\alpha_2}(g(s),t_2)}\right)\Delta s \le -\int_{t_2}^{t}(t-s)^r\omega^{\Delta}(s)\,\Delta s.$$

Since

$$-\int_{t_2}^{t}(t-s)^r\omega^{\Delta}(s)\,\Delta s = (t-t_2)^r\omega(t_2) + \int_{t_2}^{t}((t-s)^r)^{\Delta_s}\omega^{\sigma}(s)\,\Delta s.$$

Since $t \ge \sigma(t)$ and $r \ge 1$, we have

$$((t-s)^r)^{\Delta_s} = \frac{1}{\mu(s)}[(t-\sigma(s))^r - (t-s)^r] = -\frac{1}{\sigma(s)-t}[(t-s)^r - (t-\sigma(s))^r].$$

From the inequality, we obtain

$$A^{\alpha} - B^{\alpha} \ge \alpha B^{\alpha-1}(A-B).$$

Thus, for $t \ge \sigma(t)$, we have

$$[(t-s)^r - (t-\sigma(s))^r] \ge r(t-\sigma(s))^{r-1}(\sigma(s)-s).$$

Thus,

$$((t-s)^r)^{\Delta_s} \le -r(t-\sigma(s))^{r-1}.$$

Thus,

$$-\int_{t_2}^{t}(t-s)^r\omega^{\Delta}(s)\,\Delta s \le (t-t_2)^r\omega(t_2) - r\int_{t_2}^{t}(t-\sigma(s))^{r-1}\omega^{\sigma}(s)\,\Delta s.$$

Then, we have

$$\frac{1}{t^r}\int_{t_2}^{t}(t-s)^r\left(K\beta(s)Q(s) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left(\beta^{\Delta}(s)\right)_+^{\alpha+1} b^{\alpha_2}(g(s))}{\left(\beta(s)\right)^{\alpha}\delta_1^{\alpha_2}(g(s),t_2)}\right)\Delta s$$
$$\le \left(\frac{t-t_2}{t}\right)^r\omega(t_2) - \frac{r}{t^r}\int_{t_2}^{t}(t-s)^r\omega^{\sigma}(s)\,\Delta s.$$

Hence,

$$\frac{1}{t^r}\int_{t_2}^{t}(t-s)^r\left(K\beta(s)Q(s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left(\beta^{\Delta}(s)\right)_+^{\alpha+1}b^{\alpha_2}(g(s))}{(\beta(s))^{\alpha}\delta_1^{\alpha_2}(g(s),t_2)}\right)\Delta s\leq\left(\frac{t-t_2}{t}\right)^r\omega(t_2).$$

Hence,

$$\limsup_{t\to\infty}\frac{1}{t^r}\int_{t_2}^{t}(t-s)^r\left(K\beta(s)Q(s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left(\beta^{\Delta}(s)\right)_+^{\alpha+1}b^{\alpha_2}(g(s))}{(\beta(s))^{\alpha}\delta_1^{\alpha_2}(g(s),t_2)}?\right)\Delta s\leq\omega(n_2),$$

which is contrary to (2.21). The proof is completed.■

Corollary 2.5. If $\mathbb{T}=\mathbb{N}$, then (2.21) becomes

$$\limsup_{n\to\infty}\frac{1}{n^r}\sum_{s=n_0}^{n-1}(n-s)^r\left(K\rho(s)Q(s)-\frac{1}{(\alpha+1)^{\alpha+1}}\frac{\left(\rho^{\Delta}(s)\right)_+^{\alpha+1}b^{\alpha_2}(g(s))}{(\rho(s))^{\alpha}\delta_1^{\alpha_2}(g(s),n_2)}\right)=\infty. \quad (\overline{2.21})$$

Then equation (1.2) has no solution of the type (I).

Now, by using the inequality
$x^{\alpha}-y^{\alpha}\geq 2^{1-\alpha}(x-y)^{\alpha}$ *for all* $x\geq y>0$ *and* $\alpha\geq 1$,

Theorem 2.4. Let $(A_1)-(A_4)$ and (S_2) hold. Further, assume that there exists a positive rd-continuous $\Delta-$ differentiable non-decreasing function $\beta(t)$, such that

$$\limsup_{t\to\infty}\frac{1}{t^r}\int_{t_2}^{t}(t-s)^r\left(K\beta(s)Q(s)-\frac{1}{2^{3-\alpha}}\frac{\left(\beta^{\Delta}(s)\right)_+^{2}b^{\alpha_2}(g(s))}{\left(\mu(g(s))\right)^{\alpha-1}\beta(s)\delta_1^{\alpha_2}(g(s),t_2)}?\right)\Delta s. \quad (2.21)$$

Then the equation (1.1) has no solution of type (I).

Proof. Let $x(t)$ be an eventually positive solution of equation (1.1) of type (I). Then, there is a $t_0\in N$ such that (I) holds for $t\geq t_0$. Define the function $\omega(t)$ by

$$\omega(t):=\beta(t)\frac{a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}}{x^{\alpha}(g(t))}. \quad (2.22)$$

Then $\omega(t)>0$. From (2.22), we have

$$\omega^{\Delta}(t)\leq\beta^{\Delta}(t)\frac{a^{\sigma}(((b(x^{\Delta})^{\alpha_1})^{\Delta})^{\sigma})^{\alpha_2}}{x^{\alpha}(g^{\sigma}(t))}+\beta(t)\frac{\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta}}{x^{\alpha}(g(t))}$$

$$-\beta(t)\frac{a^{\sigma}(((b(x^{\Delta})^{\alpha_1})^{\Delta})^{\sigma})^{\alpha_2}\left(x^{\alpha}(g(t))\right)^{\Delta}}{x^{\alpha}(g^{\sigma}(t))x^{\alpha}(g(t))}.$$

Now, by using the inequality
$x^{\alpha}-y^{\alpha}\geq 2^{1-\alpha}(x-y)^{\alpha}$ *for all* $x\geq y>0$ *and* $\alpha\geq 1$,
then, we have

$$\left(x^{\alpha}(g(t))\right)^{\Delta}=\frac{x^{\alpha}(g^{\sigma}(t))-x^{\alpha}(g(t))}{\mu(g(t))}\geq\frac{2^{1-\alpha}}{\mu(g(t))}\left(x(g^{\sigma}(t))-x(g(t))\right)^{\alpha}$$

$$= 2^{1-\alpha}\left(\mu(g(t))\right)^{\alpha-1}\left(\frac{x(g^{\sigma}(t)) - x(g(t))}{\mu(g(t))}\right)^{\alpha}$$

$$= 2^{1-\alpha}\left(\mu(g(t))\right)^{\alpha-1}\left(x^{\Delta}(g(t))\right)^{\alpha}, \alpha \geq 1.$$

Thus

$$\omega^{\Delta}(t) \leq \beta^{\Delta}(t)\frac{a^{\sigma}(((b(x^{\Delta})^{\alpha_1})^{\Delta})^{\sigma})^{\alpha_2}}{x^{\alpha}(g^{\sigma}(t))} + \beta(t)\frac{\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta}}{x^{\alpha}(g(t))}$$
$$-2^{1-\alpha}\left(\mu(g(t))\right)^{\alpha-1}\beta(t)\frac{a^{\sigma}(((b(x^{\Delta})^{\alpha_1})^{\Delta})^{\sigma})^{\alpha_2}\left(x^{\Delta}(g(t))\right)^{\alpha}}{x^{\alpha}(g^{\sigma}(t))x^{\alpha}(g(t))}. \quad (2.23)$$

From *Lemma* 2.1, there exists $t_3 \geq t_2$ with $g(t) \geq t_2$ for all $t \geq t_3$ such that

$$\left(x^{\Delta}(g(t))\right)^{\alpha} \geq b^{-\alpha_2}(g(t))\left(a(g(t))\left(\left(b(g(t))\left(x^{\Delta}(g(t))\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)\left(\int_{t_2}^{g(t)} a^{-\frac{1}{\alpha_2}}(s)\,\Delta s\right)^{\alpha_2}.$$

Since $\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta} \leq 0, g(t) < t$, we get

$$a(\sigma(t))\left(\left(b(\sigma(t))\left(x^{\Delta}(\sigma(t))\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2} \leq a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}$$
$$\leq a(g(t))\left(\left(b(g(t))\left(x^{\Delta}(g(t))\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}.$$

From(2.23) and the above inequality, we obtain

$$\omega^{\Delta}(t) \leq \frac{\beta^{\Delta}(t)}{\beta^{\sigma}(t)}\omega^{\sigma}(t) + \beta(t)\frac{\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta}}{x^{\alpha}(g(t))}$$
$$-2^{1-\alpha}\left(\mu(g(t))\right)^{\alpha-1}\frac{\beta(t)}{\left(\beta^{\sigma}(t)\right)^{2}b^{\alpha_2}(g(t))}\delta_1^{\alpha_2}(g(t),t_2)\left(\omega^{\sigma}(t)\right)^{2}. \quad (2.24)$$

By using (1.1), (2.24), (S_2) and the inequality $Bu - Au^2 \leq \frac{B^2}{4A}$, $A > 0$ in (2.24), we have

$$\omega^{\Delta}(t) \leq -K\beta(t)Q(t)$$
$$+\frac{1}{2^{3-\alpha}}\frac{\left(\beta^{\Delta}(t)\right)_+^{2}b^{\alpha_2}(g(t))}{\left(\mu(g(t))\right)^{\alpha-1}\beta(t)\delta_1^{\alpha_2}(g(t),t_2)}. \quad (2.25)$$

From (2.25) for $t \geq t_2$, we obtain

$$\int_{t_2}^{t}(t-s)^r\left(K\beta(s)Q(s) - \frac{1}{2^{3-\alpha}}\frac{\left(\beta^{\Delta}(s)\right)_+^{2}b^{\alpha_2}(g(s))}{\left(\mu(g(s))\right)^{\alpha-1}\beta(s)\delta_1^{\alpha_2}(g(s),t_2)}\right)\Delta s \leq -\int_{t_2}^{t}(t-s)^r\omega^{\Delta}(s)\,\Delta s.$$

Since

$$((t-s)^r)^{\Delta_s} \leq -r(t-\sigma(s))^{r-1}.$$

Thus,

$$-\int_{t_2}^{t}(t-s)^r\omega^{\Delta}(s)\,\Delta s = (t-t_2)^r\omega(t_2) - r\int_{t_2}^{t}(t-s)^{r-1}\omega^{\sigma}(s)\,\Delta s.$$

Then, we have

$$\frac{1}{t^r}\int_{t_2}^{t}(t-s)^r\left(K\beta(s)Q(s)-\frac{1}{2^{3-\alpha}}\frac{\left(\beta^{\Delta}(s)\right)_+^2 b^{\alpha_2}(g(s))}{\left(\mu(g(s))\right)^{\alpha-1}\beta(s)\delta_1^{\alpha_2}(g(s),t_2)}\right)\Delta s$$
$$\leq\left(\frac{t-t_2}{t}\right)^r\omega(t_2)-\frac{r}{t^r}\int_{t_2}^{t}(t-s)^r\omega^{\sigma}(s)\,\Delta s.$$

Hence,

$$\frac{1}{t^r}\int_{t_2}^{t}(t-s)^r\left(K\beta(s)Q(s)-\frac{1}{2^{3-\alpha}}\frac{\left(\beta^{\Delta}(s)\right)_+^2 b^{\alpha_2}(g(s))}{\left(\mu(g(s))\right)^{\alpha-1}\beta(s)\delta_1^{\alpha_2}(g(s),t_2)}\right)\Delta s\leq\left(\frac{t-t_2}{t}\right)^r\omega(t_2).$$

We get,

$$\limsup_{t\to\infty}\frac{1}{t^r}\int_{t_2}^{t}(t-s)^r\left(K\beta(s)Q(s)-\frac{1}{2^{3-\alpha}}\frac{\left(\beta^{\Delta}(s)\right)_+^2 b^{\alpha_2}(g(s))}{\left(\mu(g(s))\right)^{\alpha-1}\beta(s)\delta_1^{\alpha_2}(g(s),t_2)}\right)\Delta s\leq\omega(t_2),$$

which is contrary to (2.21). This completes the proof of Theorem 2.4. ■

Corollary 2.6. If $\mathbb{T}=\mathbb{N}$, then (2.19) becomes

$$\limsup_{n\to\infty}\frac{1}{n^r}\sum_{s=n_0}^{n-1}(n-s)^r\left(K\rho(s)Q(s)-\frac{1}{2^{3-\alpha}\rho(s)}\frac{\left(\Delta\rho(s)\right)^2 b^{\alpha_2}(g(s))}{\delta_1^{\alpha_2}(g(s),n_2)}\right)$$
$$=\infty. \qquad (2.21)$$

Then equation (1.2) has no solution of the type (I).

2.2. Nonexistence of solutions of type (II)

Next, we shall establish some criteria for the nonexistence of solution of type (II) for equation (1.1).
Theorem 2.5. Assume that $(A_1)-(A_4)$ and (S_1) hold, and there exist two functions $\xi(n)$ *and* $\eta(n)$ such that

$$\Delta\xi(t)\geq 0,\xi(t)>t \text{ and } \eta(t)=g\left(\xi(\xi(t))\right)<t. \qquad (2.26)$$

If the first order delay equation

$$x^{\Delta}(t)+f^{\frac{1}{\alpha}}\left(x(\eta(t))\right)b^{-\frac{1}{\alpha_1}}(t)\left(\int_{t}^{\xi(t)}a^{-\frac{1}{\alpha_2}}(u)\left(\int_{t}^{\xi(u)}Q(s)\,\Delta s\right)^{\frac{1}{\alpha_2}}\Delta u\right)^{\frac{1}{\alpha_1}}, \qquad (2.27)$$

is oscillatory, then equation (1.1) has no solution of the type (II).
Proof. Let $x(t)$ be an eventually positive solution of the equation (1.1) of the type (II). Then, there is a $t_0\in N$ such that type (II) holds for $t\geq t_0$. By integrating equation (1.1) from t to $\xi(t)$, we obtain

$$a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\geq\int_{t}^{\xi(t)}Q(s)f\left(x(g(s))\right)\Delta s.$$

Using (2.26) and (S_1), we get

$$\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta} \geq a^{-\frac{1}{\alpha_2}}(t) f^{\frac{1}{\alpha_2}}\left(x\left(g(\xi(t))\right)\right)\left(\int_{t}^{\xi(t)} Q(s)\,\Delta s\right)^{\frac{1}{\alpha_2}}.$$

Integrating again the above inequality from t to $\xi(t)$, we find

$$-b(t)\left(x^{\Delta}(t)\right)^{\alpha_1} \geq \int_{t}^{\xi(t)} a^{-\frac{1}{\alpha_2}}(u) f^{\frac{1}{\alpha_2}}\left(x\left(g(\xi(u))\right)\right)\left(\int_{t}^{\xi(u)} Q(s)\,\Delta s\right)^{\frac{1}{\alpha_2}} \Delta u.$$

It follows that

$$-x^{\Delta}(t) \geq f^{\frac{1}{\alpha}}\left(x(\eta(t))\right) b^{-\frac{1}{\alpha_1}}(t)\left(\int_{t}^{\xi(t)} a^{-\frac{1}{\alpha_2}}(u)\left(\int_{t}^{\xi(u)} Q(s)\,\Delta s\right)^{\frac{1}{\alpha_2}} \Delta u\right)^{\frac{1}{\alpha_1}}.$$

Finally, integrating the above inequality from t *to* ∞, we have

$$x(t) \geq f^{\frac{1}{\alpha}}\left(x(\eta(t))\right) \int_{t}^{\infty}\left(b^{-\frac{1}{\alpha_1}}(v)\left(\int_{t}^{\xi(v)} a^{-\frac{1}{\alpha_2}}(u)\left(\int_{t}^{\xi(u)} Q(s)\,\Delta s\right)^{\frac{1}{\alpha_2}} \Delta u\right)^{\frac{1}{\alpha_1}}\right)\Delta v.$$

The function $x(t)$ is obviously decreasing strictly . Hence, by the discrete analog of Theorem 1 in [14], we conclude that there exists a positive solution of equation (2.27) which tends to zero. This contradicts that (2.27) is oscillatory. The proof is complete.∎

Corollary 2.7. If $\mathbb{T} = \mathbb{N}$, then (2.26) and (2.27) becomes

$$\Delta\xi(n) \geq 0, \xi(n) > n \text{ and } \eta(n) = g\left(\xi(\xi(n))\right) < n. \qquad (\overline{2.26})$$

If the first order delay equation

$$\Delta(x_n) + b^{-\frac{1}{\alpha_1}}(n) f^{\frac{1}{\alpha}}\left(x(\eta(n))\right)\left(\sum_{s=n}^{\xi(n)-1} a^{-\frac{1}{\alpha_2}}(s)\left(\sum_{u=n}^{\xi(s)-1} Q(u)\right)^{\frac{1}{\alpha_2}}\right)^{\frac{1}{\alpha_1}} = 0, \qquad (\overline{2.27})$$

is oscillatory, then equation (1.2) has no solution of the type (II).

2.3. Oscillation criteria under condition (2.1)

Next, we shall establish some oscillation criteria for equation (1.1) under condition (2.1).

Theorem 2.6. Let (2.1), (2.5) *and* (J_1) hold, where (J_1), (S_1) *and* (2.9)hold. Then equation (1.1) is oscillatory or $\lim_{n\to\infty} x(n) = 0$.

Proof. On the contrary, assuming that (1.1) has a non-oscillatory solution, then, without loss of generality, there is a $t_1 \geq t_0$ such that $x(t) > 0$ *and* $x(g(t)) > 0$. From the proof of Lemma 2.2 $x(t)$ is either of type (I) or (II). From Theorem (2.1), $x(t)$ is not of type (I). From Lemma (2.3), we have $\lim_{t\to\infty} x(t) = 0$. The proof is complete.∎

The proof of each of the following corollary is similar to that of Theorem 2.6 and hence the details are omitted.

Corollary 2.8. Let $(2.1),(2.5)$ $and(\mathrm{J}_2)$ hold. Where $(\mathrm{J}_2),(S_3)$ $and(2.10)$ hold. Then equation (1.1) is oscillatory or $\lim_{t\to\infty} x(t) = 0$.
Remark 2.1. If $\alpha_1 \equiv 1, n \equiv 1$. Then Corollary 2.8 reduced to a special case of Theorem 1 in [5].
Remark 2.2. Corollary 2.8 extended and improved Theorem 2.1 in [8].
Remark 2.3. If $a(t) \equiv b(t) \equiv 1, \alpha_1 \equiv \alpha_2 \equiv 1, n \equiv 1$. Then Corollary 2.8 reduced to a special case of Theorem 2.6 in [6].
Corollary 2.9. Let $(2.1),(2.5)$ $and(\mathrm{J}_3)$ hold. Where $(\mathrm{J}_3),(S_3)$ $and(2.19)$ hold. Then equation (1.1) is oscillatory or $\lim_{t\to\infty} x(t) = 0$.
Remark 2.4. If $\alpha_1 \equiv 1, n \equiv 1$. Then Corollary 2.9 reduced to a special case of Theorem 5 in [5].
Corollary 2.10. Let $(2.1),(2.5)$ $and(\mathrm{J}_4)$ hold. Where $(\mathrm{J}_4),(S_2)$ $and(2.21)$ hold. Then equation (1.1) is oscillatory or $\lim_{t\to\infty} x(t) = 0$.
Remark 2.5. If $\alpha_1 \equiv 1, n \equiv 1$ Corollary 2.10 extended and improved Theorem 4 in [5].

Theorem 2.7. Let (2.1) holds, and there exist two functions $\xi(t)$ and $\eta(t)$ such that (2.26) and (2.27) hold. Assume that (J_1) holds. Then equation (1.1) is oscillatory.
Proof. On the contrary, assuming that (1.1) has a non-oscillatory solution, then, without loss of generality, there is a $t_1 \geq t_0$ such that $x(t) > 0$ and $x\big(g(t)\big) > 0$. From the proof of Lemma 2.2 $x(t)$ is either of type (I) or (II). From Theorem (2.1), $x(t)$ is not of type (I). From Theorem (2.5), $x(t)$ is not of type (II). The proof is complete.■

The proof of each of the following corollary is similar to that of Theorem 2.7 and hence the details are omitted.

Corollary 2.11. Let (2.1) holds, and there exist two functions $\xi(t)$ and $\eta(t)$ such that (2.26) and (2.27) hold. Assume that (J_2) holds. Then equation (1.1) is oscillatory.
Corollary 2.12. Let (2.1) holds, and there exist two functions $\xi(t)$ and $\eta(t)$ such that (2.26) and (2.27) hold. Assume that (J_3) holds. Then equation (1.1) is oscillatory.
Corollary 2.13. Let (2.1) holds, and there exist two functions $\xi(t)$ and $\eta(t)$ such that (2.26) and (2.27) hold. Assume that (J_4) holds. Then equation (1.1) is oscillatory.

2.4. Nonexistence of solutions of type (III)

Next, we shall establish some criteria for the nonexistence of solution of type (III) for equation (1.1).

Theorem 2.8. Assume that $(A_1)-(A_3)$ and (S_1) hold, if the first order delay equation

$$\int_{t_0}^{\infty}\left(a^{-\frac{1}{\alpha_2}}(s)\left(\int_{t_0}^{s} Q(r) f\left(\int_{t_0}^{g(r)} b^{-\frac{1}{\alpha_1}}(v)\,\Delta v\right) f\left(\int_{g(r)}^{\infty} a^{-\frac{1}{\alpha_2}}(k)\,\Delta k\right)^{\frac{1}{\alpha_1}} \Delta r\right)^{\frac{1}{\alpha_2}}\right)\Delta s = \infty, \qquad (2.28)$$

is oscillatory, then equation (1.1) has no solution of type (III).
Proof. Let $x(t)$ be an eventually positive solution of equation (1.1) of type (III). Then, there is a $t \in [t_0,\infty)_{\mathbb{T}}$ such that (III) holds for $t \geq t_0$. Then, we have

$$x(t) - x(t_3) = \int_{t_3}^{t} x^{\Delta}(s)\,\Delta s = \int_{t_3}^{t} b^{-\frac{1}{\alpha_1}}(s)\left(b(s)\left(x^{\Delta}(s)\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}} \Delta s$$

$$\geq \left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}} \int_{t_3}^{t} b^{-\frac{1}{\alpha_1}}(s)\,\Delta s, \qquad for\ t \geq t_3,$$

and hence

$x(t) \geq \left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}} \int_{t_3}^{t} b^{-\frac{1}{\alpha_1}}(s)\,\Delta s,$ for $t \geq t_3$.

There exists a $t_4 \geq t_3$ with $g(t) \geq t_3$ for all $t \geq t_4$, such that

$$x(g(t)) \geq \left(b(g(t))\left(x^{\Delta}(g(t))\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}} \int_{t_3}^{g(t)} b^{-\frac{1}{\alpha_1}}(s)\,\Delta s, \qquad for\ t \geq t_4.$$

From equation(1.1), (S_1) and the last inequality, we obtain, for $t \geq t_4$

$$0 \geq \left(a(t)\left(v^{\Delta}(t)\right)^{\alpha_2}\right)^{\Delta} + Q(t)f\left(v^{\frac{1}{\alpha_1}}(g(t))\right)f\left(\int_{t_3}^{g(t)} b^{-\frac{1}{\alpha_1}}(s)\,\Delta s\right), \tag{2.29}$$

where $v(t) := b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}$. It is clear that $v(t) > 0$ and $\Delta v(t) < 0$. It follows that

$-a(t)\left(v^{\Delta}(t)\right)^{\alpha_2} \geq -a(t_4)\left(v^{\Delta}(t_4)\right)^{\alpha_2}$ for $t \geq t_4$.

Thus

$$-v^{\Delta}(t) \geq -\frac{a^{\frac{1}{\alpha_2}}(t_4)v^{\Delta}(t_4)}{a^{\frac{1}{\alpha_2}}(t)} \ for\ t \geq t_4.$$

Integrating the last inequality from t to ∞, we obtain

$$v(t) \geq -a^{\frac{1}{\alpha_2}}(t_4)v^{\Delta}(t_4)\int_{t}^{\infty} a^{-\frac{1}{\alpha_2}}(s)\,\Delta s = K_1 \int_{t}^{\infty} a^{-\frac{1}{\alpha_2}}(s)\,\Delta s, \quad for\ t \geq t_4,$$

where $K_1 := -a^{\frac{1}{\alpha_2}}(t_4)v^{\Delta}(t_4) > 0$. There exists a $t_5 \geq t_4$ with $g(t) \geq t_4$ for all $t \geq t_5$, such that

$$v(g(t)) \geq K_1 \int_{g(t)}^{\infty} a^{-\frac{1}{\alpha_2}}(s)\,\Delta s, \ \ for\ t \geq t_5.$$

Integrating (2.29) from t_5 to t and using the above inequality, we find

$$\int_{t_5}^{t} Q(r)f\left(\int_{t_3}^{g(r)} b^{-\frac{1}{\alpha_1}}(s)\,\Delta s\right)f\left(K_1\int_{g(r)}^{\infty} a^{-\frac{1}{\alpha_2}}(k)\,\Delta k\right)^{\frac{1}{\alpha_1}} \Delta r \leq a(t_5)\left(v^{\Delta}(t_5)\right)^{\alpha_2} - a(t)\left(v^{\Delta}(t)\right)^{\alpha_2},$$

By using (S_1), we see that

$$\left(\frac{L}{a(t)}\int_{t_5}^{t} Q(r)f\left(\int_{t_3}^{g(r)} b^{-\frac{1}{\alpha_1}}(s)\,\Delta s\right)f\left(\int_{g(r)}^{\infty} a^{-\frac{1}{\alpha_2}}(k)\,\Delta k\right)^{\frac{1}{\alpha_1}} \Delta r\right)^{\frac{1}{\alpha_2}} \leq -v^{\Delta}(t),$$

where $L := f\left(K_1^{\frac{1}{\alpha_1}}\right)$. Integrating the above inequality from t_5 to ∞, we obtain

$$L^{\frac{1}{\alpha_2}}\int_{t_5}^{\infty}\left(a^{-\frac{1}{\alpha_2}}(s)\left(\int_{t_5}^{s} Q(r)f\left(\int_{t_3}^{g(r)} b^{-\frac{1}{\alpha_1}}(v)\,\Delta v\right)f\left(\int_{g(r)}^{\infty} a^{-\frac{1}{\alpha_2}}(k)\,\Delta k\right)^{\frac{1}{\alpha_1}} \Delta r\right)^{\frac{1}{\alpha_2}}\right)\Delta s \leq v(t_5) < \infty,$$

which contradicts the condition (2.28). The proof is complete.■

Corollary 2.14. If $\mathbb{T}=\mathbb{N}$, then (2.28) becomes

$$\sum_{s=n_0}^{\infty}\left(a^{-\frac{1}{\alpha_2}}(s)\left(\sum_{r=n_0}^{s-1}Q(r)f\left(\sum_{u=n_0}^{g(r)-1}b^{-\frac{1}{\alpha_1}}(u)\right)f\left(\sum_{v=g(r)}^{\infty}a^{-\frac{1}{\alpha_2}}(v)\right)^{\frac{1}{\alpha_1}}\right)^{\frac{1}{\alpha_2}}\right)=\infty. \qquad (\overline{2.28})$$

Then equation (1.1) has no solution of type (III).

Theorem 2.9. Assume that $(A_1)-(A_3)$ hold and (S_2), let $\beta(t)$ be a positive rd-continuous $\Delta-$ differentiable function. If

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(b^{-\frac{1}{\alpha_1}}(r)\left(\int_{t_0}^{r}a^{-\frac{1}{\alpha_2}}(s)\left(\int_{t_0}^{s}\Psi(u)\left(\int_{g(u)}^{\infty}\frac{1}{a(\tau)}\Delta\tau\right)^{\alpha_2}\Delta u\right)^{\frac{1}{\alpha_2}}\Delta s\right)^{\frac{1}{\alpha_1}}\right)\Delta r=\infty. \qquad (2.30)$$

Then equation (1.1) has no solution of type (III).

Proof. Let $x(t)$ be an eventually positive solution of equation (1.1) of type (III). Then, there exists $t_2\geq t_1$ such that $x^{\Delta}(t)>0$, $\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}<0$ for all $t\geq t_2$. Then, we have

$$x^{\Delta}(t)=\frac{\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}}}{(b(t))^{\frac{1}{\alpha_1}}}.$$

Integrating the above inequality from t_2 to t, we obtain

$$x(t)-x(t_2)=\int_{t_2}^{t}\frac{\left(b(s)\left(x^{\Delta}(s)\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}}}{(b(s))^{\frac{1}{\alpha_1}}}\Delta s$$

$$\geq\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}}\int_{t_2}^{t}\frac{1}{(b(s))^{\frac{1}{\alpha_1}}}\Delta s. \qquad (2.31)$$

Hence there exists a $t_3\geq t_2$ such that

$$x(g(t))\geq\left(b(g(t))\left(x^{\Delta}(g(t))\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}}\int_{t_2}^{g(t)}\frac{1}{(b(s))^{\frac{1}{\alpha_1}}}\Delta s, for\ t\geq t_3.$$

From equation (1.1), (S_2) and the last inequality, we obtain

$$\left(a(t)\left(v^{\Delta}(t)\right)^{\alpha_2}\right)^{\Delta}+KQ(t)\left(v(g(t))\right)^{\alpha_2}\left(\int_{t_2}^{g(t)}\frac{1}{(b(s))^{\frac{1}{\alpha_1}}}\Delta s\right)^{\boldsymbol{\alpha}}\leq 0, t\geq t_3, \qquad (2.32)$$

where $v(t):=b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}$. It is clear that $v(t)>0$ and $v^{\Delta}(t)<0$. It follows that

$$\left(a(t)\left(v^{\Delta}(t)\right)^{\alpha_2}\right)^{\Delta}+\Psi(t)v^{\alpha_2}(g(t))\leq 0, \qquad for\ t\geq t_3. \qquad (2.33)$$

Since $g(t)\to\infty$ as $t\to\infty$, we can choose $t_4\geq t_3$ such that $g(t)\geq t_4$ for $t\geq t_4$ and thus

$$v(\infty) - v(g(t)) = \int_{g(t)}^{\infty} a(s)v^{\Delta}(s)\frac{1}{a(s)}\Delta s$$

$$< v^{\Delta}(g(t))a(g(t)) \int_{g(t)}^{\infty} \frac{1}{a(s)}\Delta s < a(t_4)v^{\Delta}(t_4) \int_{g(t)}^{\infty} \frac{1}{a(s)}\Delta s.$$

Thus

$$-v(g(t)) < a(t_4)v^{\Delta}(t_4) \int_{g(t)}^{\infty} \frac{1}{a(s)}\Delta s.$$

By substituting the above inequality in (2.33), we get

$$\left(a(t)\left(v^{\Delta}(t)\right)^{\alpha_2}\right)^{\Delta} < L^{\alpha_2}\Psi(t)\left(\int_{g(t)}^{\infty} \frac{1}{a(s)}\Delta s\right)^{\alpha_2}, \qquad for\ t \geq t_4, \tag{2.34}$$

where $L = a(t_4)v^{\Delta}(t_4) < 0$. Integrating this inequality from t_4 to t, we see that

$$a(t)\left(v^{\Delta}(t)\right)^{\alpha_2} < a(t)\left(v^{\Delta}(t)\right)^{\alpha_2} - a(t_4)\left(v^{\Delta}(t_4)\right)^{\alpha_2} < L^{\alpha_2} \int_{t_4}^{t} \Psi(s)\left(\int_{g(s)}^{\infty} \frac{1}{a(\tau)}\Delta\tau\right)^{\alpha_2} \Delta s.$$

where $v^{\Delta}(t) < 0$. Integrating again from t_5 to t, we have

$$v(t) < L \int_{t_5}^{t} a^{-\frac{1}{\alpha_2}}(s)\left(\int_{t_4}^{s} \Psi(u)\left(\int_{g(u)}^{\infty} \frac{1}{a(\tau)}\Delta\tau\right)^{\alpha_2} \Delta u\right)^{\frac{1}{\alpha_2}} \Delta s$$

or equivalently

$$x^{\Delta}(t) < \left(\frac{L}{b(t)}\right)^{\frac{1}{\alpha_1}}\left(\int_{t_5}^{t} a^{-\frac{1}{\alpha_2}}(s)\left(\int_{t_4}^{s} \Psi(u)\left(\int_{g(u)}^{\infty} \frac{1}{a(\tau)}\Delta\tau\right)^{\alpha_2} \Delta u\right)^{\frac{1}{\alpha_2}} \Delta s\right)^{\frac{1}{\alpha_1}}.$$

Integrating from t_6 to t, we have

$$x(t) < x(t_6) + L^{\frac{1}{\alpha_1}} \int_{t_6}^{t}\left(b^{-\frac{1}{\alpha_1}}(r)\left(\int_{t_5}^{r} a^{-\frac{1}{\alpha_2}}(s)\left(\int_{t_4}^{s} \Psi(u)\left(\int_{g(u)}^{\infty} \frac{1}{a(\tau)}\Delta\tau\right)^{\alpha_2} \Delta u\right)^{\frac{1}{\alpha_2}} \Delta s\right)^{\frac{1}{\alpha_1}}\right)\Delta r.$$

From condition (2.30), we have $\lim_{n\to\infty} x(t) = -\infty$ which contradicts the fact that $x(t) > 0$. The proof is complete . ■

Corollary 2.15. If $\mathbb{T} = \mathbb{N}$, then (2.30) becomes

$$\limsup_{n\to\infty} \sum_{u=n_0}^{n-1}\left(b^{-\frac{1}{\alpha_1}}(u)\left(\sum_{s=n_0}^{u-1} a^{-\frac{1}{\alpha_2}}(s)\left(\sum_{t=n_0}^{s-1} \Psi(t)\left(\sum_{\tau=g(t)}^{\infty} a^{-1}(\tau)\right)^{\alpha_2}\right)^{\frac{1}{\alpha_2}}\right)^{\frac{1}{\alpha_1}}\right) = \infty. \tag{$\overline{2.30}$}$$

Then equation (1.2) has no solution of type (III).

2.5 Oscillation criteria under condition (2.2)

Next, we shall establish some oscillation criteria for equation (1.1) under condition (2.2).

Theorem 2.11. Let $(2.2), (2.5)$ *and* (2.28) hold. Assume that (J_1) or (J_2) or (J_3) or (J_4) holds. Then equation (1.1) is oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Proof. To the contrary assume that (1.1) has a non-oscillatory solution. Then, without loss of generality, there is a $t_1 \geq t_0$ such that $x(t) > 0$ *and* $x(g(t)) > 0$. From (2.2), there exist three possible cases (I), (II) and (III). From Theorem (2.1) or (2.2) or (2.3) or (2.4) respectively, $x(t)$ is not of type (I). From Lemma (2.3), we have $\lim_{t\to\infty} x(t) = 0$. From Theorem (2.8), $x(t)$ is not of type (III). The proof is complete.■

Theorem 2.12. Let (2.2) *and* (2.28) hold, and there exist two functions $\xi(t)$ *and* $\eta(t)$ such that (2.26) *and* (2.27) hold. Assume that (J_1) or (J_2) or (J_3) or (J_4) holds. Then equation (1.1) is oscillatory.

Proof. To the contrary assume that (1.1) has a non-oscillatory solution. Then, without loss of generality, there is a $t_1 \geq t_0$ such that $x(t) > 0$ *and* $x(g(t)) > 0$. Then, proceeding as in the proof of theorem (2.11), we obtain $x(t)$ is not of type (I). From Theorem (2.5), $x(t)$ is not of type (II). From Theorem (2.8), $x(t)$ is not of type (III). The proof is complete.■

Theorem 2.13. Let $(2.2), (2.5)$ *and* (2.30) hold. Assume that (J_1) or (J_2) or (J_3) or (J_4) holds. Then equation (1.1) is oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Proof. To the contrary assume that (1.1) has a non-oscillatory solution. Then, without loss of generality, there is a $t_1 \geq t_0$ such that $x(t) > 0$ *and* $x(g(t)) > 0$. Then, proceeding as in the proof of theorem (2.11), we obtain $x(t)$ is not of type (I). From Lemma (2.3), we have, $\lim_{t\to\infty} x(t) = 0$. From Theorem (2.9), $x(t)$ is not of type (III). The proof is complete.■

Theorem 2.14. Let (2.2) *and* (2.30) hold, and there exist two functions $\xi(t)$ *and* $\eta(t)$ such that (2.26) *and* (2.27) hold. Assume that (J_1) or (J_2) or (J_3) or (J_4) holds. Then equation (1.1) is oscillatory .

Proof. To the contrary assume that (1.1) has a non-oscillatory solution. Then, without loss of generality, there is a $t_1 \geq t_0$ such that $x(t) > 0$ *and* $x(g(t)) > 0$. proceeding as in the proof of theorem (2.12), we obtain $x(t)$ is not of type (I)or (II). From Theorem (2.9), $x(t)$ is not of type (III). The proof is complete.■

2.6 Nonexistence of solutions of type (IV)

Next, we shall establish some criteria for the nonexistence of solution of type (IV) for equation (1.1).

Theorem 2.15. Assume that $(A_1) - (A_3)$ and (S_1) hold, if

$$\int_{t_0}^{\infty} \left(b^{-\frac{1}{\alpha_1}}(l) \left(\int_{t_0}^{l} a^{-\frac{1}{\alpha_2}}(k) \left(\int_{t_0}^{k} Q(s) f\left(\int_{g(s)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\, \Delta r \right) \Delta s \right)^{\frac{1}{\alpha_2}} \Delta k \right)^{\frac{1}{\alpha_1}} \right) \Delta l = \infty . \qquad (2.35)$$

Then equation (1.1) has no solution of type (IV).

Proof. Let $x(t)$ be an eventually positive solution of equation (1.1) of type (IV). Then, there is a $t \in [t_0, \infty)_{\mathbb{T}}$ such as that (IV) holds for $t \geq t_0$. We one can choose $t_3 \geq t_2$ with $g(t) \geq t_2$ $for\ all\ t \geq t_3$, such that

$$x(g(t)) = -\int_{g(t)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\left(b(r)\left(x^{\Delta}(r)\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}}\Delta r$$

$$\geq -\left(b(g(t))\left(x^{\Delta}(g(t))\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}}\int_{g(t)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\,\Delta r$$

$$\geq K_2\int_{g(t)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\,\Delta r\ for\ t \geq t_3,$$

where $K_2 := -\left(b(g(t))\left(x^{\Delta}(g(t))\right)^{\alpha_1}\right)^{\frac{1}{\alpha_1}} > 0$. Thus equation (1.1) and (S_1) yield

$$\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta} \leq -Q(t)f\left(x(g(t))\right)$$

$$\leq LQ(t)f\left(\int_{g(t)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\,\Delta r\right).$$

where $L := -f(K_2) < 0$. Integrating the above inequality from t_3 to t, we find

$$a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2} \leq L\int_{t_3}^{t} Q(s)f\left(\int_{g(t)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\,\Delta r\right)\Delta s.$$

Hence,

$$\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta} \leq L^{\frac{1}{\alpha_2}}a^{-\frac{1}{\alpha_2}}(t)\left(\int_{t_3}^{t} Q(s)f\left(\int_{g(s)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\,\Delta r\right)\Delta s\right)^{\frac{1}{\alpha_2}}.$$

Again Integrating the above inequality from t_3 to t, we find

$$b(t)\left(x^{\Delta}(t)\right)^{\alpha_1} \leq L^{\frac{1}{\alpha_2}}\int_{t_3}^{t} a^{-\frac{1}{\alpha_2}}(s)\left(\int_{t_3}^{s} Q(u)f\left(\int_{g(s)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\,\Delta r\right)\Delta u\right)^{\frac{1}{\alpha_2}}\Delta s.$$

It follows that

$$x^{\Delta}(t) \leq K_3 b^{-\frac{1}{\alpha_1}}(t)\left(\int_{t_3}^{t} a^{-\frac{1}{\alpha_2}}(s)\left(\int_{t_3}^{s} Q(u)f\left(\int_{g(s)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\,\Delta r\right)\Delta u\right)^{\frac{1}{\alpha_2}}\Delta s\right)^{\frac{1}{\alpha_1}}.$$

where $K_3 := L^{\frac{1}{\alpha}}$. Finally, Integrating the above inequality from t_3 to t, we have

$$x(t) \leq x(t_3) + K_3\int_{t_3}^{t}\left(b^{-\frac{1}{\alpha_1}}(s)\left(\int_{t_3}^{s} a^{-\frac{1}{\alpha_2}}(u)\left(\int_{t_3}^{u} Q(v)f\left(\int_{g(v)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\,\Delta r\right)\Delta v\right)^{\frac{1}{\alpha_2}}\Delta u\right)^{\frac{1}{\alpha_1}}\right)\Delta s.$$

From condition (2.35), we get $x(t) \to -\infty$ as $t \to \infty$, which contradicts the fact that $x(t)$ is a positive solution of (1.1). The proof is complete.■

Corollary 2.16. If $\mathbb{T} = \mathbb{N}$, then (2.35) becomes

$$\sum_{l=n_0}^{\infty}\left(b^{-\frac{1}{\alpha_1}}(l)\left(\sum_{k=n_0}^{l-1} a^{-\frac{1}{\alpha_2}}(k)\left(\sum_{s=n_0}^{k-1} Q(s) f\left(\sum_{r=g(s)}^{\infty} b^{-\frac{1}{\alpha_1}}(r)\right)\right)^{\frac{1}{\alpha_2}}\right)^{\frac{1}{\alpha_1}}\right) = \infty . \qquad (\overline{2.35})$$

Then equation (1.1) has no solution of type (IV).

Theorem 2.16. Assume that $(S_2), (A_1) - (A_3)$ hold. If

$$\int_{t_0}^{\infty}\left(b^{-\frac{1}{\alpha_1}}(u)\left(\int_{t_0}^{u} a^{-\frac{1}{\alpha_2}}(s)\,\Delta s\right)^{\frac{1}{\alpha_1}}\right)\Delta u = \infty . \qquad (2.36)$$

Then equation (1.1) has no solution of type (IV).

Proof. Let $x(t)$ be an eventually positive solution of equation (1.1) of type (IV). Then, there is a $t \in [t_0, \infty)_{\mathbb{T}}$ such that (IV) holds for $t \geq t_0$. Since $a(t)\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}$ is non-increasing function there exists a negative constant K_4 *and* $t_2 \geq t_1$ such that

$$a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2} \leq K_4 \; for \; t \geq t_2.$$

Dividing by $a(t)$ and integrating the last inequality from t_1 *to* t, we obtain

$$x^{\Delta}(t) \leq b^{-\frac{1}{\alpha_1}}(t) K_4^{\frac{1}{\alpha}}\left(\int_{t_1}^{t} a^{-\frac{1}{\alpha_2}}(s)\,\Delta s\right)^{\frac{1}{\alpha_1}} .$$

Integrating the last inequality from t_1 *to* t, we obtain

$$x(t) \leq x(t_1) + K_4^{\frac{1}{\alpha}}\int_{t_1}^{t}\left(b^{-\frac{1}{\alpha_1}}(u)\left(\int_{t_1}^{u} a^{-\frac{1}{\alpha_2}}(s)\,\Delta s\right)^{\frac{1}{\alpha_1}}\right)\Delta u.$$

Letting $t \to \infty$ then, by, (2.36) we deduce that $x(t) \to -\infty$, which is contradiction to the fact that $x(t) > 0$.

Corollary 2.17. If $\mathbb{T} = \mathbb{N}$, then, (2.36) becomes

$$\sum_{u=n_0}^{\infty}\left(b^{-\frac{1}{\alpha_1}}(u)\left(\sum_{s=n_0}^{u-1} a^{-\frac{1}{\alpha_2}}(s)\right)^{\frac{1}{\alpha_1}}\right) = \infty . \qquad (\overline{2.36})$$

Then the equation (1.1) has no solution of type (IV).

2.7. Oscillation criteria under condition (2.3)

Next, we shall establish some oscillation criteria for the equation (1.1) under condition (2.3).

Theorem 2.17. Let $(2.3), (2.5) and (2.35)$ hold. Assume that (J_1) or (J_2) or (J_3) or (J_4) holds. And $(2.28) or (2.30)$ holds. Then equation (1.1) is oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Proof. To the contrary assume that (1.1) has a non-oscillatory solution. Then, without loss of generality, there is a $n_1 \geq n_0$ such that $x(t) > 0$ *and* $x(g(t)) > 0$. From (2.3), there exist four possible cases (I) , (II) , (III) and (III). From Theorem (2.1) or (2.2) or (2.3) or (2.4) respectively, $x(t)$ is not of type (I). From Lemma (2.3), we have, $\lim_{t\to\infty} x(t) = 0$. From Theorem$(2.8) or (2.9)$ respectively, $x(t)$ is not of type (III). From Theorem (2.15), $x(t)$ is not of type (IV). The proof is complete.■

Theorem 2.18. Let (2.3) *and* (2.35) hold, and there exist two functions $\xi(t)$ *and* $\eta(t)$ such that (2.26) *and* (2.27) hold. Assume that (J_1) or (J_2) or (J_3) or (J_4) holds. And $(2.28) or (2.30)$ holds. Then equation (1.1) is oscillatory.

Proof. To the contrary assume that (1.1) has a non-oscillatory solution. Then, without loss of generality, there is a $t_1 \geq t_0$ such that $x(t) > 0$ *and* $x(g(t)) > 0$. Then, proceeding as in the proof of Theorem (2.17), we obtain $x(t)$ is not of type (I) or (III). From Theorem (2.5), $x(t)$ is not of type (II). From Theorem (2.15), $x(t)$ is not of type (IV). The proof is complete.■

Theorem 2.19. Let $(2.3), (2.5) and (2.36)$ hold. Assume that (J_1) or (J_2) or (J_3) or (J_4) holds. And $(2.28) or (2.30)$ holds. Then equation (1.1) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$.

Proof. To the contrary, assume that (1.1) has a non-oscillatory solution. Then, without loss of generality, there is a $t_1 \geq t_0$ such that $x(t) > 0$ *and* $x(g(t)) > 0$. Then, proceeding as in the proof of theorem (2.17), we obtain $x(t)$ is not of type (I) or (III). By Lemma (2.3), we get $\lim_{t\to\infty} x(t) = 0$. From Theorem (2.16), $x(t)$ is not of type (IV). The proof is complete.■

Theorem 2.20. Let (2.3) *and* (2.36) hold, and there exist two functions $\xi(t)$ *and* $\eta(t)$ such as that (2.26) *and* (2.27) hold. Assume that (J_1) or (J_2) or (J_3) or (J_4) holds. And $(2.28) or (2.30)$ holds. Then equation (1.1) is oscillatory.

Proof. To the contrary assume that (1.1) has a non-oscillatory solution. Then, without loss of generality, there is a $t_1 \geq t_0$ such that $x(t) > 0$ *and* $x(g(t)) > 0$. Then, proceeding as in the proof of Theorem (2.17), we obtain $x(t)$ is not of type (I) or (II) or (III). From Theorem (2.16), $x(t)$ is not of type (IV). The proof is complete.■

3.Examples:

In this section we will show the applications of our oscillation criteria by three examples. We will see that the equations in the example are oscillatory or tend to zero based on the results in section 2.

Example 3.1. Consider the third order delay dynamic equation

$$\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta} + \sum_{i=1}^{n} q_i(t) f\left(x(g_i(t))\right) = 0 \quad (3.1)$$

(with $a(t) \equiv b(t) \equiv 1, n = 1, \alpha_1 \equiv \alpha_2 \equiv 1, q(t) \equiv \frac{\gamma}{tg(t)}, \gamma > 0, \beta(t) = t \text{ and } f\left(x(g(t))\right) \equiv x(g(t))$.

All the conditions of corollary (2.8) are satisfied. Hence every solution of equation (3.1) is either oscillatory or satisfies $\lim_{n\to\infty} x_n = 0$.

Example 3.2. Consider the linear delay dynamic equation

$$\left(a(t)\left(\left(b(t)\left(x^{\Delta}(t)\right)^{\alpha_1}\right)^{\Delta}\right)^{\alpha_2}\right)^{\Delta}+\sum_{i=1}^{n} q_i(t)\, f\left(x\big(g_i(t)\big)\right)=0 \tag{3.2}$$

with $a(t) \equiv b(t) \equiv 1, n = 1, \alpha_1 \equiv \alpha_2 \equiv 1, q(t) \equiv \frac{27}{32}, \beta(t) = 1$ and $f\left(x\big(g(t)\big)\right) \equiv (t - 2), H(t,s) = t - s.$

All the conditions of corollary (2.9) are satisfied. Hence every solution of equation (3.2) is either oscillatory or satisfies $\lim_{n\to\infty} x_n = 0.$

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