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Development in a New Approach to Find the Determinant of Matrix

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Abstract

Omid Rezaifer and Hossein Rezaee [Appl. Math. and Comput. **188**, 1445 (2007)] presented a method of finding the determinants of a matrix in 2007. In this paper we present a new approach of this method which makes it more general. We used Choi's condensation method and determinant of minor of a matrix in this approach.

Keywords: Determinant, Matrix.

ملخص: قدم كلا من اوميد رضايفر و حسين رضائي في البحث المنشور في مجلة الرياضيات والحسابات التطبيقية طريقة لإيجاد محدد مصفوفة في ٢٠٠٧. في هذا البحث نقدم طريقة جديدة لها تجعلها أكثر عمومية. استخدمنا طريقة تشوي للتكثيف ومحدد مصفوفة صغيرة للمصفوفة في هذه الطريقة.

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1 Introduction

F. Choi, H. Eves, Jacobi, Q. Gjonbalaj, and A. Salihu have given some methods to obtain the value of determinants along with some properties of these determinants. In this paper we developed a new approach to compute the determinants of matrices of the order $n \times n$ given by Omid Rezaifer and Hossein Rezaee [6].

2 Preliminaries

In this section, some basic definitions are given and some preliminary result are proved. Moreover, the formulas given by Omid Rezaifer and Hossein Rezaee are discussed.

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Definition 2.1: Consider a $n \times n$ matrix, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$,

then the minor matrix of A is

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix},$$

where

$$m_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(j-1)} & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix}$$

Definition 2.2: Let $A = (a_{ij})_{n\times n}$ and m_{ij} be minors of elements in A, then $C = (c_{ij})_{n\times n}$ is called co-factor matrix of A, where $c_{ij} = (-1)^{i+j} m_{ij}$.
Definition 2.3: The transpose of co-factor matrix of matrix A is called adjoin of matrix A and is denoted by $adj(A)$.
Definition 2.4: If A is an $n \times n$ nonsingular square matrix then the matrix B is said to be inverse of matrix A if $AB = BA = I_n$, where I_n is a unit matrix. Thus if A^{-1} is inverse of A, then

$$AA^{-1} = I_n. \tag{1}$$

Also

$$A^{-1} = \frac{1}{Det(A)} adj(A) \tag{2}$$

Theorem 2.1 (Choi's Theorem):([1],[3],[5],[6])
Let A be $m \times m$ matrix as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}$$

Then,

$$DetA = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix}$$

$$= \frac{1}{a_{11}^2} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \dots & \begin{vmatrix} a_{11} & a_{1m} \\ a_{21} & a_{2m} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & \dots & \begin{vmatrix} a_{11} & a_{1m} \\ a_{31} & a_{3m} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{m1} & a_{m2} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{m1} & a_{m3} \end{vmatrix} & \dots & \begin{vmatrix} a_{11} & a_{1m} \\ a_{m1} & a_{mm} \end{vmatrix} \end{vmatrix}$$

Theorem 2.2(Omid and Rezaee): [6]
Consider $A = (a_{ij})$ be an $n \times n$ matrix, then

$$Det(A) = \frac{\begin{vmatrix} |M_{11}| & |M_{1n}| \\ |M_{n1}| & |M_{nn}| \end{vmatrix}}{|M_{11,nn}|} \qquad if\ M_{11,nn} \neq 0, \tag{3}$$

where

$$M_{ij} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(j-1)} & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{bmatrix},$$

$$M_{11,nn} = \begin{bmatrix} a_{22} & a_{23} & \cdots & a_{2(n-1)} \\ a_{32} & a_{33} & \cdots & a_{3(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n-1)2} & a_{(n-1)3} & \cdots & a_{(n-1)(n-1)} \end{bmatrix}.$$

This formula is the same as Jacobean's formula [4][8],[7].
Theorem 2.3(Jacobean): Let $A = (a_{ij})$ be an $n \times n$ matrix, then

$$Det \begin{bmatrix} a'_{11} & a'_{1n} \\ a'_{n1} & a'_{nn} \end{bmatrix} = Det(intA).DetA,$$

where

$$a'_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(j-1)} & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(j-1)} & a_{n(j+1)} & \cdots & a_{nn} \end{vmatrix},$$

and $int(A) = \begin{bmatrix} a_{22} & a_{23} & \cdots & a_{2(n-1)} \\ a_{32} & a_{33} & \cdots & a_{3(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n-1)2} & a_{(n-1)3} & \cdots & a_{(n-1)(n-1)} \end{bmatrix}$. Now we prove the following results.

Theorem 2.4: Consider $A = [a_{ij}]$ be an $n \times n$ matrix, and M is a minor matrix of a matrix A, then

$$Det(M) = Det(co(A)) = Det(adj(A)) = (Det(A))^{n-1},$$

where

$$M = \begin{vmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{vmatrix}$$

Proof: We have

$$Det(co(A)) = Det(adj(A))$$

.

From the equation (1) and (2) we get

$A\frac{1}{Det(A)}adj(A) = I \Rightarrow Det(A\frac{1}{Det(A)}adj(A)) = Det(I)$

$\Rightarrow DetA\frac{1}{(DetA)^n}Det(adj(A)) = 1 \Rightarrow \frac{1}{(DetA)^{n-1}}Det(adj(A)) = 1$

$\Rightarrow Det(adj(A)) = (DetA)^{n-1}$. But $cof(A) = (adj(A))^T$, therefore

$$Det(cof(A)) = Det(adj(A)) = (Det(A))^{n-1}. \tag{4}$$

Now we prove that, $Det(M) = Det(cof(A))$.
Since

$$\begin{aligned} Det(M) = \sum_{j=1}(-1)^{i+j}m_{ij}M_{ij} &= \sum_{j=1} c_{ij} \times (-1)^{i+j}C_{ij} \\ &= \sum_{j=1}(-1)^{(i+j)}c_{ij}C_{ij} \\ &= \sum_{j=1}(-1)^{(i+j)}c_{ij}C_{ij} \\ &= Det(cof(A)). \end{aligned}$$

By using the equation (4) we get,

$$DetM = Det(cof(A)) = Det(adj(A)) = (DetA)^{n-1} \tag{5}$$

as desired.
From the above result we get the following result:
Theorem 2.5: Let $A = (a_{ij})$ be an $n \times n$ matrix and M is its minor matrix, then the determinant of minor matrix of $M(a_{ij})$ is equal m_{ij}^{n-2}.
Proof: Since

$$M(a_{ij}) = [m_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2(j-1)} & a_{2(j+1)} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{i(j-1)} & a_{i(j+1)} & \cdots & a_{nn} \end{bmatrix}$$

Here, $M(a_{ij})$ is of order $(n-1)$, therefore using Theorem 2.4, we obtain

$$Det(M(M(a_{ij}))) = (Det(M(a_{ij})))^{n-2}$$

But $DetM(a_{ij}) = m_{ij}$, therefore $Det(M(M(a_{ij}))) = m_{ij}^{n-2}$ as desired.

3 Main Result

In this section we present a new algorithm to compute the determinant of a matrix. This theorem depends on the int matrix, minor matrix or both of them. The following result is the generalization of the new approach given in ([6]).
Theorem 3.1: Let $A = (a_{ij})$ be an $n \times n$ matrix and M is its minor matrix, then

$$Det(A) = \frac{\begin{vmatrix} m_{ij} & m_{ik} \\ m_{lj} & m_{lk} \end{vmatrix}}{m_{ij,lk}}$$

where $m_{ij,lk} \neq 0$.
Proof: By using Choi's method we get,

$$Det(M) = \begin{vmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{vmatrix}$$

$$= \frac{1}{m_{11}^{n-2}} \begin{vmatrix} \begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{13} \\ m_{21} & m_{23} \end{vmatrix} & \cdots & \begin{vmatrix} m_{11} & m_{1n} \\ m_{21} & m_{2n} \end{vmatrix} \\ \begin{vmatrix} m_{11} & m_{12} \\ m_{31} & m_{32} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{vmatrix} & \cdots & \begin{vmatrix} m_{11} & m_{1n} \\ m_{31} & m_{3n} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{vmatrix} m_{11} & m_{12} \\ m_{n1} & m_{22} \end{vmatrix} & \begin{vmatrix} m_{11} & m_{13} \\ m_{n1} & m_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} m_{11} & m_{1n} \\ m_{n1} & m_{nn} \end{vmatrix} \end{vmatrix}$$

where $m_{11} \neq 0$. Suppose that

$$Det(A) \neq \frac{\begin{vmatrix} m_{ij} & m_{ik} \\ m_{lj} & m_{lk} \end{vmatrix}}{m_{ij,lk}}$$

where

$$m_{ij,lk} \neq 0.$$

Put $i, j = 1, l \in \{2, 3, ..., n\}$ and $k \in \{2, 3, ..., n\}$, then

$$Det(M) \neq \frac{1}{m_{11}^{n-2}} \begin{vmatrix} Det(A)m_{11,22} & Det(A)m_{11,23} & \cdots & Det(A)m_{11,2n} \\ Det(A)m_{11,32} & Det(A)m_{11,33} & \cdots & Det(A)m_{11,3n} \\ \vdots & \vdots & \ddots & \vdots \\ Det(A)m_{11,n2} & Det(A)m_{11,n3} & \cdots & Det(A)m_{11,nn} \end{vmatrix}$$

$$Det(M) \neq \frac{Det(A)^{n-1}}{m_{11}^{n-2}} \begin{vmatrix} m_{11,22} & m_{11,23} & \cdots & m_{11,2n} \\ m_{11,32} & m_{11,33} & \cdots & m_{11,3n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{11,n2} & m_{11,n3} & \cdots & m_{11,nn} \end{vmatrix}.$$

Therefore by using Theorem 1.5, we get

$$Det(M) \neq Det(A)^{n-1}$$

which is a contradiction. Thus

$$Det(A) = \frac{\begin{vmatrix} m_{ij} & m_{ik} \\ m_{lj} & m_{lk} \end{vmatrix}}{m_{ij,lk}}.$$

Theorem 3.2: If $m_{ij,lk} = 0$, in the Theorem 3.1, then

$$\begin{vmatrix} m_{ij} & m_{ik} \\ m_{lj} & m_{lk} \end{vmatrix} = 0$$

Proof: By using Theorem 3.1, we get

$$m_{ij} = \frac{1}{m_{ij,(i+1)(j+1),(l-1)(k-1),lk}} \begin{vmatrix} m_{ij,(i+1)(j+1)} & m_{ij,(i+1)k} \\ m_{ij,l(j+1)} & m_{ij,lk} \end{vmatrix} \tag{6}$$

$$m_{ik} = \frac{1}{m_{ij,(i+1)(j+1),(l-1)(k-1),lk}} \begin{vmatrix} m_{ik,(i+1)j} & m_{ik,(i+1)(k-1)} \\ m_{ik,lj} & m_{ik,l(k-1)} \end{vmatrix} \tag{7}$$

$$m_{lj} = \frac{1}{m_{ij,(i+1)(j+1),(l-1)(k-1),lk}} \begin{vmatrix} m_{lj,i(j+1)} & m_{lj,ik} \\ m_{lj,(l-1)(j+1)} & m_{lj,(l-1)k} \end{vmatrix} \tag{8}$$

and

$$m_{lk} = \frac{1}{m_{ij,(i+1)(j+1),(l-1)(k-1),lk}} \begin{vmatrix} m_{lk,ij} & m_{lk,l(k-1)} \\ m_{lk,(l-1)j} & m_{lk,(l-1)(k-1)} \end{vmatrix}. \tag{9}$$

Therefore

$$\begin{vmatrix} m_{ij} & m_{ik} \\ m_{lj} & m_{lk} \end{vmatrix} = \frac{1}{m^2_{ij,(i+1)(j+1),(l-1)(k-1),lk}} \\ \big(m_{ij,(i+1)(j+1)}m_{ij,lk}m_{lk,ij}m_{lk,(l-1)(k-1)} \\ - m_{ij,(i+1)(j+1)}m_{ij,lk}m_{lk,(l-1)j}m_{lk,i(k-1)} \\ - m_{ij,(i+1)k}m_{ij,l(j+1)}m_{lk,ij}m_{lk,(l-1)(k-1)} \\ + m_{ij,(i+1)k}m_{ij,l(j+1)}m_{lk,(l-1)j}m_{lk,i(k-1)} \\ - m_{ik,(i+1)j}m_{ik,l(k-1)}m_{lj,i(j+1)}m_{lj,(l-1)k} \\ + m_{ik,(i+1)j}m_{ik,l(k-1)}m_{lj,ik}m_{lj,(l-1)(j+1)} \\ + m_{ik,lj}m_{ik,(i+1)(k-1)}m_{lj,i(j+1)}m_{lj,(l-1)k} \\ - m_{ik,lj}m_{ik,(i+1)(k-1)}m_{lj,ik}m_{lj,(l-1)(j+1)}\big).$$

But we observe that,

$$m_{ij,lk} = m_{lk,ij} = m_{ik,lj} = m_{lj,ik}. \tag{10}$$

Therefore

$$\begin{vmatrix} m_{ij} & m_{ik} \\ m_{lj} & m_{lk} \end{vmatrix} = \frac{1}{m^2_{ij,(i+1)(j+1),(l-1)(k-1),lk}} \\ \big(m_{ij,(i+1)(j+1)}m^2_{ij,lk}m_{lk,(l-1)(k-1)} \\ - m_{ij,(i+1)(j+1)}m_{ij,lk}m_{lk,(l-1)j}m_{lk,i(k-1)} \\ - m_{ij,(i+1)k}m_{ij,l(j+1)}m_{ij,lk}m_{lk,(l-1)(k-1)} \\ + m_{ij,(i+1)k}m_{ij,l(j+1)}m_{lk,(l-1)j}m_{lk,i(k-1)} \\ - m_{ik,(i+1)j}m_{ik,l(k-1)}m_{lj,i(j+1)}m_{lj,(l-1)k} \\ + m_{ik,(i+1)j}m_{ik,l(k-1)}m_{ij,lk}m_{lj,(l-1)(j+1)} \\ + m_{ij,lk}m_{ik,(i+1)(k-1)}m_{lj,i(j+1)}m_{lj,(l-1)k} \\ - m^2_{ij,lk}m_{ik,(i+1)(k-1)}m_{lj,(l-1)(j+1)}\big)$$

But $m_{ij,lk} = 0$, therefore

$$\begin{vmatrix} m_{ij} & m_{ik} \\ m_{lj} & m_{lk} \end{vmatrix} = \frac{1}{m^2_{ij,(i+1)(j+1),(l-1)(k-1),lk}} \\ \big(m_{ij,(i+1)k}m_{ij,l(j+1)}m_{lk,(l-1)j}m_{lk,i(k-1)} \\ - m_{ik,(i+1)j}m_{ik,l(k-1)}m_{lj,i(j+1)}m_{lj,(l-1)k}\big).$$

Since $m_{ij,(i+1)k} = m_{ik,(i+1)j}$, $m_{ij,l(j+1)} = m_{lj,l(j+1)}$, $m_{lk,(l-1)j} = m_{lj,(l-1)k}$
and $m_{lk,i(k-1)} = m_{ik,l(k-1)}$, so that

$$m_{ij,(i+1)k}m_{ij,l(j+1)}m_{lk,(l-1)j}m_{lk,i(k-1)} = m_{ik,(i+1)j}m_{ik,l(k-1)}m_{lj,i(j+1)}m_{lj,(l-1)k}.$$

Thus

$$\begin{vmatrix} m_{ij} & m_{ik} \\ m_{lj} & m_{lk} \end{vmatrix} = 0$$

as desired.

Corollary 3.1: Let $A=(a_{ij})_{n\times n}$ and if $m_{11,nn}=0$, then

$$\begin{vmatrix} m_{11} & m_{1n} \\ m_{n1} & m_{nn} \end{vmatrix} = 0.$$

Theorem 3.2 Consider $A=(a_{ij})$ be an $n\times n$ matrix, then

$$\lim_{m_{ij,lk}\to 0} \frac{\begin{vmatrix} m_{ij} & m_{ik} \\ m_{lj} & m_{lk} \end{vmatrix}}{m_{ij,lk}} = Det(A).$$

Proof: Considering Theorem 3.1, it is enough to prove that,

$$Det(A) = \lim_{m_{ij,lk}\to 0} \frac{\begin{vmatrix} m_{ij} & m_{ik} \\ m_{lj} & m_{lk} \end{vmatrix}}{m_{ij,lk}}$$

is exist. By using (6), (7), (8) and (9), we get

$$\begin{aligned}\lim_{m_{ij,lk}\to 0} \begin{vmatrix} m_{ij} & m_{ik} \\ m_{lj} & m_{lk} \end{vmatrix} = \lim_{m_{ij,lk}\to 0} \frac{1}{m_{ij,lk} m^2_{ij,(i+1)(j+1),(l-1)(k-1),lk}} \\ (m_{ij,(i+1)(j+1)} m_{ij,lk} m_{lk,ij} m_{lk,(l-1)(k-1)} \\ - m_{ij,(i+1)(j+1)} m_{ij,lk} m_{lk,(l-1)j} m_{lk,i(k-1)} \\ - m_{ij,(i+1)k} m_{ij,l(j+1)} m_{lk,ij} m_{lk,(l-1)(k-1)} \\ + m_{ij,(i+1)k} m_{ij,l(j+1)} m_{lk,(l-1)j} m_{lk,i(k-1)} \\ - m_{ik,(i+1)j} m_{ik,l(k-1)} m_{lj,i(j+1)} m_{lj,(l-1)k} \\ + m_{ik,(i+1)j} m_{ik,l(k-1)} m_{lj,ik} m_{lj,(l-1)(j+1)} \\ + m_{ik,lj} m_{ik,(i+1)(k-1)} m_{lj,i(j+1)} m_{lj,(l-1)k} \\ - m_{ik,lj} m_{ik,(i+1)(k-1)} m_{lj,ik} m_{lj,(l-1)(j+1)}).\end{aligned}$$

Using (10), we get

$$\begin{aligned}&\lim_{m_{ij,lk}\to 0} \frac{1}{m_{ij,lk}} \begin{vmatrix} m_{ij} & m_{ik} \\ m_{lj} & m_{lk} \end{vmatrix} \\ &= \lim_{m_{ij,lk}\to 0} \frac{1}{m_{ij,lk} m^2_{ij,(i+1)(j+1),(l-1)(k-1),lk}} \\ &(m_{ij,(i+1)(j+1)} m^2_{ij,lk} m_{lk,(l-1)(k-1)} \\ &- m_{ij,(i+1)(j+1)} m_{ij,lk} m_{lk,(l-1)j} m_{lk,i(k-1)} \\ &- m_{ij,(i+1)k} m_{ij,l(j+1)} m_{ij,lk} m_{lk,(l-1)(k-1)} \\ &+ m_{ij,(i+1)k} m_{ij,l(j+1)} m_{lk,(l-1)j} m_{lk,i(k-1)} \\ &- m_{ik,(i+1)j} m_{ik,l(k-1)} m_{lj,i(j+1)} m_{lj,(l-1)k} \\ &+ m_{ik,(i+1)j} m_{ik,l(k-1)} m_{ij,lk} m_{lj,(l-1)(j+1)} \\ &+ m_{ij,lk} m_{ik,(i+1)(k-1)} m_{lj,i(j+1)} m_{lj,(l-1)k} \\ &- m^2_{ij,lk} m_{ik,(i+1)(k-1)} m_{lj,(l-1)(j+1)}).\end{aligned}$$

Since $m_{ij,(i+1)k} = m_{ik,(i+1)j}$, $m_{ij,l(j+1)} = m_{lj,l(j+1)}$, $m_{lk,(l-1)j} = m_{lj,(l-1)k}$ and $m_{lk,i(k-1)} = m_{ik,l(k-1)}$, therefore

$$m_{ij,(i+1)k} m_{ij,l(j+1)} m_{lk,(l-1)j} m_{lk,i(k-1)} = m_{ik,(i+1)j} m_{ik,l(k-1)} m_{lj,i(j+1)} m_{lj,(l-1)k}.$$

Thus

$$\lim_{m_{ij,lk}\to 0}\frac{1}{m_{ij,lk}}\begin{vmatrix} m_{ij} & m_{ik} \\ m_{lj} & m_{lk}\end{vmatrix} = \frac{1}{m^2_{ij,(i+1)(j+1),(l-1)(k-1),lk}}(-m_{ij,(i+1)(j+1)}m_{lk,(l-1)j}m_{lk,i(k-1)}$$
$$- m_{ij,(i+1)k}m_{ij,l(j+1)}m_{lk,(l-1)(k-1)}$$
$$+ m_{ik,(i+1)j}m_{ik,l(k-1)}m_{lj,(l-1)(j+1)}$$
$$+ m_{ik,(i+1)(k-1)}m_{lj,i(j+1)}m_{lj,(l-1)k}).$$

Therefore

$$\lim_{m_{ij,lk}\to 0}\frac{1}{m_{ij,lk}}\begin{vmatrix} m_{ij} & m_{ik} \\ m_{lj} & m_{lk}\end{vmatrix}$$

exists as desired.

Corollary 3.2: Consider $A = (a_{ij})$ be an $n \times n$ matrix, then

$$\lim_{m_{11,nn}\to 0}\frac{1}{m_{11,nn}}\begin{vmatrix} m_{11} & m_{1n} \\ m_{n1} & m_{nn}\end{vmatrix} = Det(A). \tag{11}$$

4 Conclusion

In this work we generalized the method given by Omid Rezaiver and Hossein Rezaee [6].

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